Math 680 Fall 2017

Fifth Homework Assignment

Exercise 14a. Show that

$$\Gamma(s) = \lim_{n\to\infty} \frac{n^s n!}{s(s+1)(s+2)\cdots(s+n)}$$

for all $s \neq 0, -1, -2, \ldots$

b) Show that $\Gamma(s+1) = s\Gamma(s)$ for all $s \neq 0, -1, -2, \ldots$

c) Show that $\Gamma(1) = 1$ and, more generally, $\Gamma(n+1) = n!$ for all non-negative integers n.

Exercise 15. Show that

$$\lim_{N\to\infty} \prod_{n=1}^{N} \frac{(2n)^2}{(2n-1)(2n+1)} = \frac{\pi}{2},$$

and thus

$$\lim_{n\to\infty} \frac{2^{2n}(n!)^2}{(2n+1)!}\sqrt{2n+1} = \sqrt{\pi/2}.$$

Hint: we proved an infinite product representation for the sine function, and you know the value of $\sin(\pi/2)$.

Exercise 16a. Show that

$$\Gamma(s)\Gamma(1-s) = \frac{\pi}{\sin(\pi s)}$$

for all $s \notin \mathbb{Z}$. Hint: Use exercise 14a.

b) Show that

$$\Gamma(s)\Gamma\big(s + (1/2)\big) = \sqrt{\pi}2^{1-2s}\Gamma(2s).$$

For what values of s is this valid? Hint: use exercises 14a and 15.

Exercise 17. Show that

$$\zeta(s) = 2(2\pi)^{s-1}\sin(\pi s/2)\Gamma(1-s)\zeta(1-s).$$

Hint: use the functional equation we proved in class together with the exercises above.