Exercise 18. Use the functional equation

$$\zeta(s) = 2^s \pi^{s-1} \zeta(1-s) \Gamma(1-s) \sin(\pi s/2)$$

to show that $\zeta(0) = -1/2$.

Exercise 19. a) Set $F(s) = (s-1)\zeta(s)$ and write

$$F(s) = \sum_{n=0}^{\infty} a_n (s-1)^n.$$  

Using the functional equation for zeta as above, show that

$$a_1 = F'(1) = -\Gamma'(1) + 2\zeta'(0) + \log(2\pi).$$

b. Use the infinite product formulation for $\Gamma$ to show that

$$\frac{\Gamma'(1)}{\Gamma(1)} = -\gamma,$$

where $\gamma$ is Euler’s constant. Thus $\Gamma'(1) = -\gamma$.

c. Show that

$$\sum_{n \leq x} \frac{1}{n} = \log x + \gamma + O(1/x)$$

for all $x \geq 1$.

d. Using the identity

$$\zeta(s) = \sum_{n \leq x} \frac{1}{n^s} + \frac{x^{1-s}}{s-1} - s \int_x^\infty (u-[u]) u^{-s-1} \, du$$

(valid when $x$ is a positive integer), show that

$$\zeta(s) = \frac{1}{s-1} + \gamma + \sum_{n \geq 1} b_n (s-1)^n.$$  

Conclude that $a_1 = \gamma$ and $\zeta'(0) = \frac{-1}{2} \log(2\pi)$. 