Math 680 Fall 2017

Seventh Homework Assignment

Exercise 20: Assume $x \geq T \geq 2$. When $\sigma = 1 + 1/\log x$, show that

$$\log x \left[1 + \frac{x \log x}{T}\right] + \frac{(4x)^\sigma}{T(\sigma - 1)} \ll \frac{x(\log x)^2}{T}$$

and

$$\frac{\log T}{T} x^\sigma (\sigma - \sigma') \ll \frac{x(\log x)^2}{T}$$

for all $0 < \sigma' < 1$.

Exercise 21: Show that $x^{c/\log T} = T$ when $T = \exp(\sqrt{c \log x})$, and for this value of T and $\sigma' = 1 - c/\log T$

$$\begin{aligned} x^{\sigma'} (\log T)^2 + \frac{x(\log x)^2}{T} &\ll x(\log x)^2 \left(x^{-c/\log T} + T^{-1}\right) \\ &\ll \frac{x}{\exp(c\sqrt{\log x})}. \end{aligned}$$

Conclude that

$$\Psi(x) = x + O\left(\frac{x}{\exp(c\sqrt{\log x})}\right).$$

Exercise 22: Show that the limit of curvilinear integrals

$$\lim_{T \to \infty} \frac{1}{2\pi i} \int_{\sigma_0 - iT}^{\sigma_0 + iT} \frac{y^s}{s} \, ds = \begin{cases} 1 & \text{if } y > 1, \\ 1/2 & \text{if } y = 1, \\ 0 & \text{if } 0 < y < 1 \end{cases}$$

for all $\sigma_0 > 0$. In fact,

$$\frac{1}{2\pi i} \int_{\sigma_0 - iT}^{\sigma_0 + iT} \frac{y^s}{s} \, ds = \begin{cases} 1 + O(y^{\sigma_0}/T) & \text{if } y \geq 2, \\ O(y^{\sigma_0}/T) & \text{if } 0 < y \leq 1/2. \end{cases}$$

Hint: For the case $y \geq 2$ use Cauchy's theorem on rectangles containing the origin that stretch further and further to the left but remain fixed at real part equal to σ_0 on the right.