Exercise 4. a) In class and in the notes we stated the definition of the limit superior of a sequence of real numbers. That definition was for a finite lim sup. Give the definitions for \( \limsup_{n \to \infty} x_n = \pm \infty \).

b) State the definition for the limit inferior of a sequence and the analog of the above for an infinite lim inf.

c) Let \( \{x_n\} \) be a real-valued sequence. Prove that both \( \liminf_{n \to \infty} x_n \) and \( \limsup_{n \to \infty} x_n \) exist (given the expanded definitions above), and that \( \lim_{n \to \infty} x_n \) exists if and only if the liminf and limsup are equal (in which case the limit is equal to the liminf). You may assume that any bounded set of real numbers has a least upper bound.

Exercise 5. We proved in class that \( \Theta(x) > \alpha \log x \pi(x) - \alpha x^\alpha \log x \) for all \( \alpha \in (0, 1) \). Use this to prove that

\[
\limsup \frac{\Theta(x)}{x} \geq \limsup \frac{\pi(x)}{x/\log x} \quad \text{and} \quad \liminf \frac{\Theta(x)}{x} \geq \liminf \frac{\pi(x)}{x/\log x}.
\]

Exercise 6. Show that

\[
\int_1^{x+1} \log t \, dt \geq \sum_{1 \leq n \leq x} \log n \geq \int_1^x \log t \, dt - (x - \lfloor x \rfloor) \log x
\]

for all \( x \geq 1 \).