Math 680 Fall 2017
The Prime Number Theorem

Our goal is to prove the Prime Number Theorem, which is simply expressed as

$$\lim_{x \to \infty} \frac{\pi(x)}{x} = 1.$$  

(We’ll look at more precise statements later.) Recall that we previously proved that

$$\limsup_{x \to \infty} \frac{\Psi(x)}{x} = \limsup_{x \to \infty} \frac{\pi(x)}{x/\log x},$$

$$\liminf_{x \to \infty} \frac{\Psi(x)}{x} = \liminf_{x \to \infty} \frac{\pi(x)}{x/\log x}.$$  

Thus, for our purposes it will suffice to show that

$$\lim_{x \to \infty} \frac{\Psi(x)}{x} = 1.$$  

Towards that end, we have $\Psi(x) = \sum_{n \leq x} \Lambda(x)$. In other words, $\Psi(x)$ is the “$A(x)$” for the Dirichlet series $\sum_{n \geq 1} \Lambda(n)n^{-s}$. Via the Euler product, we’ve seen that this is the Dirichlet series for $-\zeta'(s)/\zeta(s)$, and has abscissa of convergence $\sigma_c = 1$. Now using Theorem 2 from the handout on Dirichlet series, we have

$$-\frac{\zeta'(s)}{\zeta(s)} = s \int_1^\infty \Psi(u)u^{-(s+1)} \, du$$

for all $s = \sigma + it$ with $\sigma > 1$. Via the change of variables $u = e^x$, we have

(1) $$-\frac{\zeta'(s)}{s\zeta(s)} = \int_0^\infty \Psi(e^x)e^{-sx} \, dx,$$

again for all $s$ with $\sigma > 1$.

At the end of the Euler product handout we discussed the analytic properties of the function on the left side of (1). Specifically, it has a simple pole (with residue 1) at $s = 1$. Any other poles must come from zeros of the zeta function.

**Theorem** (Hademard and de la Vallée Poussin): We have $\zeta(1 + it) \neq 0$ for all $t \neq 0$.

**Corollary:** The function

$$-\frac{\zeta'(s)}{s\zeta(s)}$$

is analytic on the set $\{s = \sigma + it: \sigma \geq 1\}$, with the sole exception of a simple pole at $s = 1$ of residue 1.

We’ll prove the theorem of Hademard and de la Vallée Poussin shortly, but first see the connection to the Prime Number Theorem.
**Theorem** (Wiener-Ikehara): Suppose $A(x) \geq 0$ is non-decreasing on $[0, \infty)$ and

$$f(s) := \int_0^\infty A(x)e^{-sx} \, dx$$

is analytic on the set of $s$ with $\Re(s) \geq 1$ except for a simple pole of residue 1 at $s = 1$. Then

$$\lim_{x \to \infty} A(x)e^{-x} = 1.$$

**Corollary** (Prime Number Theorem): We have

$$\lim_{x \to \infty} \frac{\pi(x)}{x/\log x} = \lim_{x \to \infty} \frac{\Psi(x)}{e^x} = \lim_{x \to \infty} \frac{\Psi(x)}{e^x} = 1.$$ 

We postpone the proof of this theorem for the time being, but note that ultimately it turns out that the Prime Number Theorem follows (via the Euler product and more generic analysis) from the fact that the zeta function is non-vanishing on the set of $s$ with $\Re(s) \geq 1$. In fact, the converse holds as well.

**Proposition 1:** Assuming that

$$\lim_{x \to \infty} \frac{\pi(x)}{x/\log x} = 1,$$

we have $\zeta(\sigma + it) \neq 0$ for all $\sigma \geq 1$.

Proof: Let $\epsilon > 0$. By the hypothesis, we have

$$\lim_{x \to \infty} \frac{\Psi(x)}{x} = 1,$$

too, so that $|\Psi(x) - x| < \epsilon x$ for all $x \geq x(\epsilon)$ (a bound depending on $\epsilon$). Set

$$\Phi(s) := -\frac{\zeta'(s)}{s\zeta(s)} - \frac{1}{s-1}.$$

By (1) we have

$$\Phi(s) = \int_1^{x(\epsilon)} \frac{\Psi(x) - x}{x^{s+1}} \, dx$$

for all $s$ with $\Re(s) > 1$. Further, $\Phi$ is analytic on the half-plane $\{s = \sigma + it: \sigma \geq 1\}$ except for any zeros of the zeta function that occur in this half-plane. As long as $\Re(s) = \sigma > 1$, we have

$$|\Phi(s)| \leq \int_1^{x(\epsilon)} \frac{|\Psi(x) - x|}{x^2} \, dx + \int_{x(\epsilon)}^\infty \frac{\epsilon}{x^\sigma} \, dx$$

$$< c(\epsilon) + \epsilon \int_1^\infty x^{-\sigma} \, dx$$

$$= c(\epsilon) + \epsilon(\sigma - 1)^{-1},$$

where the $c(\epsilon)$ depends only on $\epsilon$. Hence $|\Phi(\sigma + it)|(\sigma - 1) \leq c(\epsilon)(\sigma - 1) + \epsilon$ for all $\epsilon > 0$ and $\sigma > 1$, implying that

$$\lim_{\sigma \to 1^+} |\Phi(\sigma + it)|(\sigma - 1) = 0.$$
for all $t$. In particular, $\Phi(s)$ has no poles on the line $s = 1 + it$ except at $s = 1$, so that $\zeta(s)$ has no zeros on this line.

We now prove the theorem of Hadamard and de la Vallée Poussin. From the Euler product we get

$$\log \zeta(s) = \log \left( \prod_{p \text{ prime}} (1 - p^{-s})^{-1} \right)$$

$$= -\sum_{p \text{ prime}} \log(1 - p^{-s})$$

$$= \sum_{p \text{ prime}} \sum_{m \geq 1} \frac{1}{mp^{-ms}}$$

$$= \sum_{n \geq 2} \frac{a_n}{n^s}$$

for all $s = \sigma + it$ with $\sigma > 1$, where

$$a_n = \begin{cases} \frac{1}{k} & \text{if } n = p^k \text{ for some prime } p, \\ 0 & \text{otherwise}. \end{cases}$$

Note that the coefficients of this Dirichlet series are non-negative. Using this series expansion, we have

$$\log |\zeta(s)| = \Re \left( \log \zeta(s) \right) = \Re \left( \sum_{n \geq 2} \frac{a_n}{n^s} \right)$$

$$= \sum_{n \geq 2} \frac{a_n \cos(t \log n)}{n^\sigma}$$

for $s = \sigma + it$ with $\sigma > 1$. In particular,

$$\log |\zeta^3(\sigma)\zeta^4(\sigma + it)\zeta(\sigma + 2it)| = 3 \log |\zeta(\sigma)| + 4 \log |\zeta(\sigma + it)| + \log |\zeta(\sigma + 2it)|$$

$$= \sum_{n \geq 2} \frac{a_n}{n^\sigma} (3 + 4 \cos(t \log n) + \cos(2t \log n)).$$

Via the trigonometric identity

$$3 + 4 \cos \theta + \cos 2\theta = 2(1 + \cos \theta)^2 \geq 0,$$

we see that the terms of the series here are all non-negative, so that $|\zeta^3(\sigma)\zeta^4(\sigma + it)\zeta(\sigma + 2it)| \geq 1$ for all $\sigma > 1$. Dividing by $\sigma - 1$ yields

$$\sigma - 1)^3|\zeta^3(\sigma)| \frac{|\zeta^4(\sigma + it)|}{(\sigma - 1)^4} |\zeta(\sigma + 2it)| \geq \frac{1}{\sigma - 1}$$

for all $\sigma > 1$.

Now suppose $\zeta(1 + it_0) = 0$ for some $t_0 \neq 0$. We know that

$$\lim_{\sigma \to 1^+} (\sigma - 1)^3|\zeta^3(\sigma)| = 1.$$
Also, since $\zeta(s)$ is analytic on the set $\{s = \sigma + it: \sigma \geq 1\} \setminus \{1\}$,

(4) \[ \lim_{\sigma \to 1^+} |\zeta(\sigma + 2it_0)| = |\zeta(1 + 2it_0)| \]

and

(5) \[ \lim_{\sigma \to 1^+} \frac{|\zeta^4(\sigma + it_0)|}{(\sigma - 1)^4} = \left| \lim_{\sigma \to 1^+} \frac{\zeta(\sigma + it_0)}{\sigma - 1} \right|^4 = |\zeta'(1 + 2it_0)|^4. \]

But taken together, (3)-(5) contradict (2), so that $\zeta(1 + it) \neq 0$ for all $t \neq 0$.

Before proving the theorem of Wiener and Ikehara, we need a few auxiliary results.

**Lemma 1:** Let $A(x)$ and $f(s)$ be as in the statement of the theorem of Wiener and Ikehara. Set $B(x) = e^{-x}A(x)$ and

\[ g(s) = \begin{cases} f(s) - \frac{1}{s-1} & \text{if } s \neq 1, \\ 0 & \text{if } s = 1. \end{cases} \]

Then $g(s)$ is analytic on $\{s = \sigma + it: \sigma \geq 1\}$ and for all $s$ with $\Re(s) > 1$,

\[ g(s) = \int_0^\infty (B(x) - 1)e^{-(s-1)x} \, dx. \]

This integral converges uniformly on any compact subset of $\{s = \sigma + it: \sigma > 1\}$.

**Proof:** The analytic property of $g$ follows from the hypotheses on $f$. For all $s$ with $\Re(s) > 1$,

\[ \int_0^\infty B(x)e^{-(s-1)x} \, dx = \int_0^\infty A(x)e^{-sx} \, dx = f(s) \]

and

\[ \int_0^\infty e^{-(s-1)x} \, dx = \left. -e^{-(s-1)x} \right|_0^\infty = \frac{1}{s-1}. \]

Let $C$ be a compact subset of $\{s = \sigma + it: \sigma > 1\}$. Then there is a $\delta > 1$ such that $\Re(s) \geq \delta$ for all $s \in C$. Let $\epsilon > 0$. By what we have already shown,

\[ \int_0^\infty B(x)e^{-(\delta-1)x} \, dx = f(\delta) \]

\[ \int_0^\infty e^{-(\delta-1)x} \, dx = \frac{1}{\delta-1}, \]

so that there is a $c > 0$ such that

\[ \max\left\{ \int_M^\infty B(x)e^{-(\delta-1)x} \, dx, \int_M^\infty e^{-(\delta-1)x} \, dx \right\} < \epsilon/2 \]
for all $M \geq c$. Now for all $s \in C$ we have

$$
\left| \int_M^\infty (B(x) - 1)e^{-(s-1)x} \, dx \right| \leq \int_M^\infty \left| (B(x) - 1)e^{-(s-1)x} \right| \, dx \\
\leq \int_M^\infty \left| (B(x) - 1) \right| e^{-(s-1)x} \, dx \\
\leq \int_M^\infty B(x)e^{-(s-1)x} \, dx + \int_M^\infty e^{-(s-1)x} \, dx \\
< \epsilon.
$$

**Lemma 2:** Suppose $\theta$ is a non-zero real number. Then for all $b > 0$

$$
\frac{1}{2} \int_{-2b}^{2b} e^{i\theta t} \left( 1 - \frac{|t|}{2b} \right) \, dt = \frac{\sin^2(b\theta)}{b\theta^2}.
$$

Proof: Since $e^{i\theta t} = \cos(\theta t) + i\sin(\theta t)$, cosine is an even function and sine is an odd function,

$$
\frac{1}{2} \int_{-2b}^{2b} e^{i\theta t} \left( 1 - \frac{|t|}{2b} \right) \, dt = \int_0^{2b} \cos(\theta t) \left( 1 - \frac{t}{2b} \right) \, dt.
$$

We integrate by parts using $u = 1 - t/2b$ and $dv = \cos(\theta t) \, dt$, which yields

$$
\int_0^{2b} \cos(\theta t) \left( 1 - \frac{t}{2b} \right) \, dt = \left( 1 - \frac{t}{2b} \right) \frac{\sin(\theta t)}{\theta} \bigg|_{t=0}^{2b} + \frac{1}{2b\theta} \int_0^{2b} \sin(\theta t) \, dt
$$

$$
= \left. -\frac{\cos(\theta t)}{2b\theta^2} \right|_{t=0}^{2b} = 1 - \cos(2b\theta) \frac{1}{2b\theta^2} = \frac{\sin^2(b\theta)}{b\theta^2}.
$$

**Lemma 3:** Suppose $h(t) : \mathbb{R} \rightarrow \mathbb{C}$ and there is a $c \geq 0$ such that $h(t) = 0$ for all $t$ with $|t| \geq c$. Then

$$
\lim_{y \to \pm\infty} \int_{-\infty}^{\infty} h(t)e^{ity} \, dt = 0.
$$

Proof: Under the change of variables $t = u + \pi/y$ we have $du = dt$ and

$$
\int_{-\infty}^{\infty} h(t)e^{ity} \, dt = \int_{-\infty}^{\infty} h(u + \pi/y)e^{iy(u+\pi/y)} \, du = -\int_{-\infty}^{\infty} h(u + \pi/y)e^{iyu} \, du,
$$

so that

$$
2 \int_{-\infty}^{\infty} h(t)e^{ity} \, dt = \int_{-\infty}^{\infty} (h(t) - h(t + \pi/y))e^{ity} \, dt.
$$

Let $\epsilon > 0$. Since $h(t)$ is continuous, it is uniformly continuous on $[-c - \pi, c + \pi]$. Thus, there is a bound $c(\epsilon) > 0$, depending only on $\epsilon$, such that $|h(t) - h(t + \pi/y)| < \epsilon/(c + \pi)$ for all $t \in [-c - \pi, c + \pi]$ and $|y| > c(\epsilon)$.
We may assume, without loss of generality, that $c(\epsilon) \geq 1$. But then both $h(t)$ and $h(t + \pi/y)$ equal 0 for all $|t| \geq c + \pi$ and $|y| > c(\epsilon)$. This shows that

$$\left| \int_{-\infty}^{\infty} h(t) e^{ity} dt \right| = \frac{1}{2} \left| \int_{-\infty}^{\infty} (h(t) - h(t + \pi/y)) e^{ity} dt \right| \leq \frac{1}{2} \int_{-\infty}^{\infty} |h(t) - h(t + \pi/y)| dt$$

< \frac{1}{2} \int_{-c-\pi}^{c+\pi} \epsilon/(c + \pi) dt = \epsilon$$

for all $|y| > c(\epsilon)$. Since $\epsilon > 0$ was arbitrary, this completes the proof.

**Lemma 4**: Suppose $F(x, y)$ is a non-negative function on $[0, \infty) \times [0, b]$ for some positive $b$, $F(x, y) \to F(x, 0)$ as $y \to 0^+$ uniformly for all $x$ in any compact subset, and $\int_{0}^{\infty} F(x, y) dx$ converges uniformly for all $y \in [0, b]$. Then

$$\lim_{y \to 0^+} \int_{0}^{\infty} F(x, y) dx = \int_{0}^{\infty} F(x, 0) dx.$$

Proof: Let $\epsilon > 0$. By hypothesis, there is a positive $c$ such that

$$\int_{c}^{\infty} F(x, y) dx < \frac{\epsilon}{4}$$

for all $y \in [0, b]$. Also, there is a $\delta > 0$ such that for all positive $y \leq \delta$ and all $x \in [0, c]$

$$|F(x, 0) - F(x, y)| < \frac{\epsilon}{2c}.$$

Therefore, for all positive $y \leq \delta$ we have

$$\left| \int_{0}^{\infty} F(x, 0) dx - \int_{0}^{\infty} F(x, y) dx \right| \leq \int_{0}^{c} |F(x, 0) - F(x, y)| dx + \int_{c}^{\infty} F(x, 0) dx + \int_{c}^{\infty} F(x, y) dx$$

$$< \int_{0}^{c} \frac{\epsilon}{2c} dx + \frac{\epsilon}{2}$$

$$< \frac{\epsilon}{2c} + \frac{\epsilon}{2}$$

$$= \epsilon.$$

**Lemma 5**: For all positive $b$

$$\lim_{y \to \infty} \int_{0}^{\infty} \frac{\sin^2 (b(y - x))}{b(y - x)^2} dx = \int_{-\infty}^{\infty} \frac{\sin^2 u}{u^2} du = \pi.$$

Proof: With the change of variables $u = b(y - x)$ we have

$$\lim_{y \to \infty} \int_{0}^{\infty} \frac{\sin^2 (b(y - x))}{b(y - x)^2} dx = \lim_{y \to \infty} \int_{by}^{-\infty} \frac{\sin^2 u}{u^2} du$$

$$= \int_{-\infty}^{\infty} \frac{\sin^2 u}{u^2} du$$

$$= \int_{0}^{\infty} \frac{1 - \cos(2u)}{u^2} du.$$
Now integrate by parts using $v = 1 - \cos(2u)$ and $dw = u^{-2} \, du$ to get

$$
\int_0^\infty \frac{1 - \cos(2u)}{u^2} \, du = \lim_{u \to 0^+} \frac{1 - \cos(2u)}{u} - \lim_{u \to \infty} \frac{1 - \cos(2u)}{u} + 2 \int_0^\infty \frac{\sin(2u)}{u} \, du
$$

$$
= 2 \int_0^\infty \frac{\sin z}{z} \, dz,
$$

where $z = 2u$. Next,

$$
\lim_{\epsilon \to 0^+} 2 \int_{\epsilon}^{\infty} \frac{\sin z}{z} \, dz = \lim_{\epsilon \to 0^+} 2 \int_{\epsilon}^{\infty} \sin z \int_0^\infty e^{-rz} \, dr \, dz.
$$

We want to change the order of integration above. Towards that end, suppose $M \geq \epsilon$. Then since

$$
\int_0^\infty e^{-rz} \sin z \, dz dr
$$

converges uniformly for all $z \in [\epsilon, M]$ we have

$$
\int_{\epsilon}^{M} \int_{\epsilon}^{\infty} e^{-rz} \sin z \, dr \, dz = \int_{\epsilon}^{\infty} \int_{\epsilon}^{M} e^{-rz} \sin z \, dz \, dr.
$$

Since

$$
\lim_{M \to \infty} \left| \int_{\epsilon}^{M} \int_{\epsilon}^{\infty} e^{-rz} \sin z \, dr \, dz \right| = \lim_{M \to \infty} \left| \int_{\epsilon}^{\infty} \int_{\epsilon}^{M} e^{-rz} \sin z \, dz \, dr \right|
$$

$$
= \lim_{M \to \infty} \left| \frac{\cos M}{M} - \int_{\epsilon}^{M} \frac{\cos z}{z^2} \, dz \right|
$$

$$
\leq \lim_{M \to \infty} \left| \frac{1}{M} + \int_{\epsilon}^{M} \frac{1}{z^2} \, dz \right|
$$

$$
= \lim_{M \to \infty} \frac{2}{M}
$$

$$
= 0,
$$

we may change the order of integration to get

$$
2 \int_{\epsilon}^{\infty} \sin z \int_{\epsilon}^{\infty} e^{-rz} \, dz \, dr = 2 \int_{\epsilon}^{\infty} \int_{\epsilon}^{\infty} e^{-rz} \sin z \, dz \, dr
$$

$$
= 2 \int_{\epsilon}^{\infty} \left( -e^{-rz} \frac{r \sin z + \cos z}{1 + r^2} \right)_{z=\epsilon}^\infty \, dr
$$

$$
= 2 \int_{\epsilon}^{\infty} e^{-\epsilon r} \frac{r \sin \epsilon + \cos \epsilon}{1 + r^2} \, dr.
$$

Note that the integrand here is non-negative and no greater than $c/(1 + r^2)$ for all $\epsilon \in [0, \pi/6]$, say, for some positive $c$. Thus these integrals are uniformly convergent for such $\epsilon$ and we may invoke Lemma 4 to get

$$
\lim_{\epsilon \to 0^+} 2 \int_{\epsilon}^{\infty} e^{-\epsilon r} \frac{r \sin \epsilon + \cos \epsilon}{1 + r^2} \, dr = 2 \int_0^\infty \frac{1}{1 + r^2} \, dr
$$

$$
= 2 \arctan r \bigg|_{r=0}^\infty
$$

$$
= \pi.
$$
**Proposition 2:** Let $A(x)$ and $f(s)$ be as in the theorem of Wiener and Ikehara. Set $B(x) = e^{-x}A(x)$ and let $b > 0$. Then
\[
\lim_{y \to \infty} \int_{-\infty}^{by} B(y - v/b) \frac{\sin^2 v}{v^2} dv = \pi.
\]

Proof: Let $g(s)$ be as in the statement of Lemma 1 and let $\delta > 0$. By Lemmas 1 and 2, for any real $y$
\[
\frac{1}{2} \int_{-2b}^{2b} g(1 + \delta + it) \left( 1 - \frac{|t|}{2b} \right) e^{iyt} dt = \frac{1}{2} \int_{-2b}^{2b} \left( 1 - \frac{|t|}{2b} \right) e^{iyt} \left( \int_{0}^{\infty} (B(x) - 1) e^{-(\delta+it)x} dx \right) dt
\]
\[
= \int_{0}^{\infty} (B(x) - 1) e^{-\delta x} \left( \int_{-2b}^{2b} \left( 1 - \frac{|t|}{2b} \right) dt \right) dx
\]
\[
= \int_{0}^{\infty} (B(x) - 1) e^{-\delta x} \frac{\sin^2 (b(y - x))}{b(y - x)^2} dx.
\]
(6)

(The changing of the order of integration above is justified since the inner integral is uniformly convergent for all $t \in [-2b, 2b]$ by Lemma 1.)

We note that since $\lim_{\delta \to 0^+} g(1 + \delta + it) = g(1 + it)$ uniformly for all $t \in [-2b, 2b]$, we may pass the limit inside the integral on the left in (6):
\[
\lim_{\delta \to 0^+} \int_{-2b}^{2b} g(1 + \delta + it) \left( 1 - \frac{|t|}{2b} \right) e^{-iyt} dt = \int_{-2b}^{2b} g(1 + it) \left( 1 - \frac{|t|}{2b} \right) e^{-iyt} dt.
\]
(7)

We claim that we may also do this on the right-hand side of (6), i.e.,
\[
\lim_{\delta \to 0^+} \int_{0}^{\infty} B(x) e^{-\delta x} \frac{\sin^2 (b(y - x))}{b(y - x)^2} dx = \int_{0}^{\infty} B(x) \frac{\sin^2 (b(y - x))}{b(y - x)^2} dx
\]
(8)

and
\[
\lim_{\delta \to 0^+} \int_{0}^{\infty} e^{-\delta x} \frac{\sin^2 (b(y - x))}{b(y - x)^2} dx = \int_{0}^{\infty} \frac{\sin^2 (b(y - x))}{b(y - x)^2} dx.
\]
(9)

To see this, we first note that
\[
0 \leq e^{-\delta x} \frac{\sin^2 (b(y - x))}{b(y - x)^2} \leq \frac{1}{b(y - x)^2}
\]
for all $\delta \geq 0$ and $\int_{0}^{\infty} \frac{1}{b(y - x)^2} dx$ converges. Thus the hypotheses of Lemma 4 are fulfilled to deduce (9).

Moreover, if we can prove the integral on the right-hand side of (8) converges, then a similar argument will give (8). By what we have already proven,
\[
\lim_{\delta \to 0^+} \int_{0}^{\infty} B(x) e^{-\delta x} \frac{\sin^2 (b(y - x))}{b(y - x)^2} dx = \frac{1}{2} \int_{-2b}^{2b} g(1 + it) \left( 1 - \frac{|t|}{2b} \right) e^{iyt} dt + \int_{0}^{\infty} \frac{\sin^2 (b(y - x))}{b(y - x)^2} dx.
\]
Denote this quantity by $M$. If the integral on the right-hand side of (8) diverges, there is a positive $c$ such that
\[
\int_{0}^{c} B(x) \frac{\sin^2 (b(y - x))}{b(y - x)^2} dx \geq 2|M|.
\]

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But now for all positive $\delta < \delta_1 = (\log 2)/c$

$$M \geq \int_0^{\infty} B(x)e^{-\delta x} \frac{\sin^2 (b(y-x))}{b(y-x)^2} \, dx \geq \int_0^{c} B(x)e^{-\delta x} \frac{\sin^2 (b(y-x))}{b(y-x)^2} \, dx$$

$$> \int_0^{c} B(x)e^{-\delta_1 c} \frac{\sin^2 (b(y-x))}{b(y-x)^2} \, dx$$

$$= \frac{1}{2} \int_0^{c} B(x) \frac{\sin^2 (b(y-x))}{b(y-x)^2} \, dx$$

$$\geq |M|.$$}

Thus the integral on the right-hand side of (8) converges, so that (8) holds as well as (9).

Combining (6)-(9) we have

$$(10) \quad \frac{1}{2} \int_{-2b}^{2b} g(1 + it) \left(1 - \frac{|t|}{2b}\right) e^{iyt} \, dt = \int_0^{\infty} B(x) \frac{\sin^2 (b(y-x))}{b(y-x)^2} \, dx - \int_0^{\infty} \frac{\sin^2 (b(y-x))}{b(y-x)^2} \, dx.$$}

This holds for all real $y$. Now set

$$h(t) = \begin{cases} \frac{1}{2} g(1 + it) \left(1 - \frac{|t|}{2b}\right) & \text{if } |t| \leq 2b, \\ 0 & \text{otherwise.} \end{cases}$$

We note that $h(t)$ satisfies the hypotheses of Lemma 3, with the $C$ there equal to $2b$ here. Applying Lemma 3 yields

$$\lim_{y \to \infty} \int_{-\infty}^{\infty} h(t) e^{iyt} \, dt = \lim_{y \to \infty} \frac{1}{2} \int_{-2b}^{2b} g(1 + it) \left(1 - \frac{|t|}{2b}\right) e^{iyt} \, dt = 0.$$}

Now by (10) and Lemma 5

$$\lim_{y \to \infty} \int_0^{\infty} B(x) \frac{\sin^2 (b(y-x))}{b(y-x)^2} \, dx = \lim_{y \to \infty} \int_0^{\infty} \frac{\sin^2 (b(y-x))}{b(y-x)^2} \, dx = \pi.$$}

The proof is completed via the change variables $v = b(y-x)$.

Proof of Wiener-Ikehara Theorem: Let $B(x) = e^{-x} A(x)$ be as above. Since both $B(x)$ and $\frac{\sin^2 v}{v^2}$ are non-negative, Proposition 2 implies that

$$\limsup_{y \to \infty} \int_{-a}^{a} B(y-v/b) \frac{\sin^2 v}{v^2} \, dv \leq \pi$$

for all positive $a$ and $b$. Since $e^x B(x) = A(x)$ is non-decreasing, for all $y > a/b$ and $|v| \leq a$ we have

$$e^{y-a/b} B(y-a/b) \leq e^{y-v/b} B(y-v/b),$$

so that

$$B(y-v/b) \geq B(y-a/b) e^{(y-a)/b} \geq B(y-a/b) e^{-2a/b}.$$
Using this, we see that
\[
\pi \geq \limsup_{y \to \infty} \int_{-a}^{a} B(y-v/b) \frac{\sin^2 v}{v^2} \, dv
\]
\[
\geq \limsup_{y \to \infty} \int_{-a}^{a} B(y-a/b)e^{-2a/b} \frac{\sin^2 v}{v^2} \, dv
\]
\[
= \limsup_{y \to \infty} B(y-a/b)e^{-2a/b} \int_{-a}^{a} \frac{\sin^2 v}{v^2} \, dv
\]
\[
= \limsup_{y \to \infty} B(y)e^{-2a/b} \int_{-a}^{a} \frac{\sin^2 v}{v^2} \, dv
\]
(setting \( b = a^2 \)).

This holds for all positive \( a \). Therefore, letting \( a \) tend to infinity and invoking Lemma 5 yields

(11) \hspace{1cm} 1 \geq \limsup_{y \to \infty} B(y) .

One consequence of (11) is that \( B(x) \leq c \) for some constant \( c \geq 0 \) and all \( x \geq 0 \). Fix \( a, b > 0 \) and assume that \( y > a/b \). We have

(12) \hspace{1cm} \int_{-\infty}^{by} B(y-v/b) \frac{\sin^2 v}{v^2} \, dv \leq c \int_{-\infty}^{a} \frac{\sin^2 v}{v^2} \, dv + c \int_{a}^{\infty} \frac{\sin^2 v}{v^2} \, dv + \int_{-a}^{a} B(y-v/b) \frac{\sin^2 v}{v^2} \, dv .

Using the fact that \( B(x)e^x \) is non-decreasing once more, we see that for all \( |v| \leq a \)

\[
B(y-v/b)e^{-v/b} \leq B(y+a/b)e^{a/v/b} ,
\]
whence

\[
B(y-v/b) \leq B(y+a/b)e^{(a+v)/b} \leq B(y+a/b)e^{2a/b} .
\]

Combining this with (12) and Proposition 2 yields

\[
\pi = \liminf_{y \to \infty} \int_{-\infty}^{by} B(y-v/b) \frac{\sin^2 v}{v^2} \, dv
\]
\[
\leq c \int_{-\infty}^{a} \frac{\sin^2 v}{v^2} \, dv + c \int_{a}^{\infty} \frac{\sin^2 v}{v^2} \, dv + \liminf_{y \to \infty} B(y+a/b)e^{2a/b} \int_{-a}^{a} \frac{\sin^2 v}{v^2} \, dv
\]
\[
\leq c \int_{-\infty}^{a} \frac{1}{v^2} \, dv + c \int_{a}^{\infty} \frac{1}{v^2} \, dv + \liminf_{y \to \infty} B(y)e^{2a/b} \int_{-a}^{a} \frac{\sin^2 v}{v^2} \, dv
\]
\[
= \frac{2c}{a} + \liminf_{y \to \infty} B(y)e^{2a/\int_{-a}^{a} \frac{\sin^2 v}{v^2} \, dv} \quad \text{(setting } b = a^2 \text{).}
\]

This is true for all positive \( a \), so letting \( a \) tend to infinity and invoking Lemma 5 gives

(13) \hspace{1cm} 1 \leq \liminf_{y \to \infty} B(y) .

Finally, combining (11) and (13) gives the desired result:

\[
1 = \lim_{y \to \infty} B(y) = \lim_{x \to \infty} A(x)e^{-x} .
\]