Math 680 Fall 2017

Basic Facts on Infinite Products

In order to see the connection between the zeta function and primes, we need to look at certain infinite products. Below are some basic definitions and results dealing with infinite products.

Definition: Suppose $a_1,\ a_2, \ldots$ is a sequence of complex numbers. We say the *infinite product* $\prod_{i=1}^{\infty} a_n$ converges to $q \neq 0$ if we have the following:

i) there is a minimal n_0 such that $a_n \neq 0$ for all $n \geq n_0$;

ii) as a generic limit of a sequence,

$$\lim_{n\to\infty} \left(\prod_{i=n_0}^{n_0+n} a_i \right) = q \neq 0.$$

Lemma 1: The infinite product $\prod a_i$ is convergent if and only if for all $\epsilon > 0$ there is an $n(\epsilon)$ such that

$$\big|(a_n a_{n+1} \cdots a_{n+k}) - 1\big| < \epsilon$$

for all $n \geq n(\epsilon)$ and all $k \geq 0$.

Proof: Suppose the product is convergent to $q \neq 0$ and let $\epsilon > 0$. Choose a positive $\delta < |q|$ with $2\delta/(|q|-\delta) < \epsilon$. There is an n_1 such that

$$|a_{n_0} \cdots a_{n_0+l} - q| < \delta$$

for all $l \geq n_1$. In particular

$$\begin{aligned} |a_{n_0} \cdots a_{n_0+l+k} - a_{n_0} \cdots a_{n_0+l}| &= |a_{n_0} \cdots a_{n_0+l+k} - q + q - a_{n_0} \cdots a_{n_0+l}| \\ &\leq |a_{n_0} \cdots a_{n_0+l+k} - q| + |q - a_{n_0} \cdots a_{n_0+l}| \\ &< 2\delta \end{aligned}$$

for all $l \geq n_1$ and $k \geq 0$. Also,

$$\begin{aligned} |a_{n_0} \cdots a_{n_0+l+k} - a_{n_0} \cdots a_{n_0+l}| &= |a_{n_0} \cdots a_{n_0+l}| \cdot |a_{n_0+l+1} \cdots a_{n_0+l+k} - 1| \\ &= |a_{n_0} \cdots a_{n_0+l} - q + q| \cdot |a_{n_0+l+1} \cdots a_{n_0+l+k} - 1| \\ &\geq \big(|q| - |a_{n_0} \cdots a_{n_0+l} - q|\big) \cdot |a_{n_0+l+1} \cdots a_{n_0+l+k} - 1| \\ &> \big(|q| - \delta\big) \cdot |a_{n_0+l+1} \cdots a_{n_0+l+k} - 1|. \end{aligned}$$

Thus

$$|a_n \cdots a_{n+k} - 1| < \frac{2\delta}{|q|-\delta} < \epsilon$$

for all $n \geq n_0 + n_1 + 1$ and all $k \geq 0$.

For the other direction, setting $\epsilon = 1/2$ shows that for some n_2 we have

$$3/2 > |a_n \cdots a_{n+k}| > 1/2$$

for all $n \geq n_2$ and all $k \geq 0$. In particular, $a_n \neq 0$ for all $n \geq n_2$ and, assuming it exists,

$$\lim_{n\to\infty} \prod_{i=n_2}^{n} a_i \neq 0.$$

Consider the sequence of "partial products" $\{p_n\}$ given by

$$p_n = \prod_{i=n_2}^{n_2+n-1} a_i$$

and let $\epsilon > 0$. Then for all n sufficiently large and all $k \geq 0$,

$$\begin{aligned}
|p_n - p_{n+k}| &= |a_{n_2} \cdots a_{n_2+n-1} - a_{n_2} \cdots a_{n_2+n+k-1}| \\
&= |a_{n_2} \cdots a_{n_2+n-2}| \cdot |a_{n_2+n-1} - a_{n_2+n-1} \cdots a_{n_2+n+k-1}| \\
&\leq |a_{n_2} \cdots a_{n_2+n-2}| \big(|a_{n_2+n-1} - 1| + |1 - a_{n_2+n-1} \cdots a_{n_2+n+k-1}| \big) \\
&< \frac{3}{2} \left(\frac{\epsilon}{3} + \frac{\epsilon}{3} \right) \\
&= \epsilon.
\end{aligned}$$

Thus, the sequence of partial products is a Cauchy sequence, whence convergent.

Remark: Setting $k = 0$ in Lemma 1, we see that $|a_n - 1| < \epsilon$ for all n sufficiently large (depending on ϵ). Thus, if the infinite product $\prod a_i$ converges, then $\lim_{n\to\infty} a_n = 1$. In other words, the infinite product

$$\prod_{i=1}^{\infty} (1 + u_n)$$

converges only if $\lim_{n\to\infty} u_n = 0$.

Definition: The infinite product $\prod(1+u_n)$ is said to be *absolutely convergent* if the product $\prod(1+|u_n|)$ is convergent.

Lemma 2: The infinite product $\prod(1 + u_n)$ is absolutely convergent if and only if the infinite sum $\sum u_n$ is absolutely convergent.

Proof: Note that for both the infinite sum $\sum |u_n|$ and infinite product $\prod(1 + |u_n|)$, convergence is just a matter of whether the (necessarily non-decreasing) sequence of partial sums or products is bounded above.

One readily verifies that the function $f(x) = e^x - x - 1$ is non-negative for all $x \geq 0$. This implies that for all $n \geq 1$

$$\begin{aligned}
|u_1| + \cdots + |u_n| &< (1 + |u_1|) \cdots (1 + |u_n|) \\
&\leq e^{|u_1|} \cdots e^{|u_n|} \\
&= e^{|u_1| + \cdots + |u_n|}.
\end{aligned}$$

Therefore, the partial sums are bounded above if and only if the partial products are bounded above.

Lemma 3: Suppose the infinite product $\prod(1+u_n)$ is absolutely convergent. Then we have the following:

i) the product $\prod(1+u_n)$ is convergent;

ii) the product $\prod(1+u_n)$ is convergent after any rearrangement of the terms;

iii) all such rearrangements yield the same limiting value.

Proof: Let $\epsilon > 0$. By Lemma 1, for all n sufficiently large and all $k \geq 0$,

$$\left|(1+|u_n|)\cdots(1+|u_{n+k}|) - 1\right| < \epsilon.$$

But

$$\left|(1+u_n)\cdots(1+u_{n+k}) - 1\right| \leq (1+|u_n|)\cdots(1+|u_{n+k}|) - 1 = \left|(1+|u_n|)\cdots(1+|u_{n+k}|) - 1\right| < \epsilon,$$

so the first part follows from Lemma 1.

Let $\sigma\colon \mathbb{N} \to \mathbb{N}$ be a rearrangement (i.e., one-to-one and onto). Since $\prod(1+u_i)$ is absolutely convergent, $\sum |u_i|$ is convergent by Lemma 2. But then $\sum |u_{\sigma(i)}|$ is also convergent (with the same sum), so that by Lemma 2 once more, $\prod(1+|u_{\sigma(i)}|)$ is convergent. By the first part of the lemma, $\prod(1+u_{\sigma(i)})$ is convergent.

For $n \geq 1$ we write $p_n = (1+u_1)\cdots(1+u_n)$ and $p'_n = (1+u_{\sigma(1)})\cdots(1+u_{\sigma(n)})$. Let $k_1 < \cdots < k_m$ denote the elements of $\{1,\ldots,n\} \setminus \{\sigma(1),\ldots,\sigma(n)\}$ arranged in increasing order and $k'_1 < \cdots < k'_l$ the elements of $\{\sigma(1),\ldots,\sigma(n)\} \setminus \{1,\ldots,n\}$ arranged in increasing order, so that

$$\frac{p_n}{p'_n} = \frac{(1+u_{k_1})\cdots(1+u_{k_m})}{(1+u_{k'_1})\cdots(1+u_{k'_l})}.$$

Considering the numerator here, we have

$$\begin{aligned}\left|(1+u_{k_1})\cdots(1+u_{k_m}) - 1\right| &\leq \big(1+|u_{k_1}|\big)\cdots\big(1+|u_{k_m}|\big) - 1\\ &\leq \exp\big(|u_{k_1}| + \cdots + |u_{k_m}|\big) - 1\\ &< \exp\left(\sum_{i=k_1}^{\infty} |u_i|\right) - 1.\end{aligned}$$

Clearly $k_1 \to \infty$ as $n \to \infty$, so by hypothesis $\sum_{i\geq k_1} |u_i| \to 0$ as $n \to \infty$. This shows that the numerator above tends to 1 as n tends to infinity. Of course, the same argument shows the denominator tends to 1 as well, proving the third part of the lemma.

Definition: A function is called *holomorphic* on a region in the complex plane if it has a derivative at every point in the region.

Lemma 4: Suppose $u_1(s),\ u_2(s),\ldots$ is a sequence of functions that are all holomorphic on an open $G \subseteq \mathbb{C}$ and $\sum |u_i(s)|$ is uniformly convergent on G with a bounded sum. Then the product $\prod 1 + u_i(s)$ is absolutely and uniformly convergent to a holomorphic function $F(s)$ on G.

Proof: If $\sum |u_i(s)|$ convergent, then by Lemma 2 $\prod \big(1+u_i(s)\big)$ is absolutely convergent, thus convergent by Lemma 3.

Suppose $\sum |u_i(s)| \leq M$ for all $s \in G$. We then have

$$\prod_{i=1}^{n} \big(1+|u_i(s)|\big) \leq \exp\left(\sum_{i=1}^{n} |u_i(s)|\right) \leq \exp(M) \tag{1}$$

for all $n \geq 1$ and all $s \in G$. Denote the partial products by $p_n(s)$. One readily sees that $p_n(s) - p_{n-1}(s) = p_{n-1}(s)u_n(s)$ for all $n \geq 2$, so that

$$\begin{aligned} p_n(s) &= p_1(s) + \sum_{i=2}^{n} p_i(s) - p_{i-1}(s) \\ &= p_1(s) + \sum_{i=2}^{n} p_{i-1}(s)u_i(s). \end{aligned}$$

Note that the sum $\sum_{i\geq 2} p_{i-1}(s)u_i(s)$ is absolutely and uniformly convergent on G by (1) and the hypotheses of the lemma. This shows that the sequence of partial products is uniformly convergent on G. Finally, as a uniform limit of holomorphic functions on G,

$$\prod_{i\geq 1} \big(1+u_i(s)\big) = \lim_{n\to\infty} p_n(s)$$

is holomorphic on G.

Lemma 5: Suppose $u_1(s),\ u_2(s), \ldots$ is a sequence of functions that are all holomorphic on an open $G \subseteq \mathbb{C}$ and $\sum |u_i(s)|$ is uniformly convergent on G with a bounded sum. Then the holomorphic function $F(s) = \prod 1 + u_i(s)$ satisfies

$$\frac{F'(s)}{F(s)} = \frac{d \log F(s)}{ds} = \sum_{i=1}^{\infty} \frac{u_i'(s)}{1+u_i(s)}$$

at every point $s \in G$ where $F(s) \neq 0$. In other words, we may differentiate the series for

$$\log F(s) = \sum_{i=1}^{\infty} \log\big(1+u_i(s)\big)$$

term-by-term.

Proof: Since the sum $\sum |u_i(s)|$ is uniformly convergent on G, there is an n_0 such that $|u_i(s)| < 1/2$ for all $i \geq n_0$ and all $s \in G$. In particular, $1 + u_i(s) \neq 0$ for all $i \geq n_0$ and all $s \in G$. Without loss of generality, we may assume that $n_0 = 1$ (since we may obviously multiply by a finite product). For $n \geq 1$ let $F_n(s) = \prod_{i\leq n} \big(1+u_i(s)\big)$ denote the partial product. Since $F_n(s) \to F(s)$ uniformly on G, we conclude that F is holomorphic on G and also that $F_n'(s) \to F'(s)$ on G.

Let $s_0 \in G$ with $F(s_0) \neq 0$. Since F is continuous, there is a non-trivial closed ball $H \subset G$ containing s_0 and a $\delta > 0$ such that $|F(s)| > \delta$ for all $s \in H$. This implies that $|F_n(s)| > \delta/2$ for all n sufficiently large and all $s \in H$. We now have

$$\frac{F'(s)}{F(s)} = \lim_{n\to\infty} \frac{F'_n(s)}{F_n(s)} = \lim_{n\to\infty} \frac{d \log F_n(s)}{ds} = \lim_{n\to\infty} \sum_{i\le n} \frac{u'_i(s)}{1+u_i(s)}$$

for all $s \in H$. In particular, this holds for $s = s_0$.