Math 680 Fall 2017

A Quantitative Prime Number Theorem I:

Zero-Free Regions

Ultimately, our goal is to prove the following strengthening of the prime number theorem.

Theorem (Improved Prime Number Theorem): There is a positive $c < 1$ such that

$$\pi(x) = \mathrm{li}(x) + O\left(\frac{x}{\exp(c\sqrt{\log x})}\right),$$

where

$$\mathrm{li}(x) = \int_2^x \frac{1}{\log x}\, dx.$$

This result will actually follow from an improved estimate on the function Ψ introduced earlier.

Theorem (Improved Estimate for Psi): There is a positive $c < 1$ such that

$$\Psi(x) = x + O\left(\frac{x}{\exp(c\sqrt{\log x})}\right).$$

Corollary: With the same constant c as above, we have

$$\Theta(x) = x + O\left(\frac{x}{\exp(c\sqrt{\log x})}\right).$$

Proof: By the definitions we have

$$\Psi(x) = \sum_{\substack{p^m \le x \\ p \text{ prime} \\ m \ge 1}} \log p = \sum_{m \ge 1} \Theta(x^{1/m}).$$

In the handout on Chebyshev's estimates we proved $\Theta(x) \le 4x \log 2$ for all $x \ge 1$. Further, Chebyshev's estimates implied that $\pi(x) \ll \frac{x}{\log x}$. Using these facts, we get

$$\begin{aligned}
\Psi(x) - \Theta(x) &= \sum_{m\ge 2} \Theta(x^{1/m}) \\
&= \Theta(x^{1/2}) + \sum_{m\ge 3} \Theta(x^{1/m}) \\
&\ll x^{1/2} + \sum_{m\ge 3} \Theta(x^{1/m}) \\
&= x^{1/2} + \sum_{m\ge 3} \sum_{\substack{p\le x^{1/m} \\ p \text{ prime}}} \log p \\
&= x^{1/2} + \sum_{\substack{p\le x^{1/3} \\ p \text{ prime}}} \log p \sum_{3\le m\le \log x/\log p} 1 \\
&< x^{1/2} + \sum_{\substack{p\le x^{1/3} \\ p \text{ prime}}} \log x \\
&= x^{1/2} + \pi(x^{1/3}) \log x \\
&\ll x^{1/2} + x^{1/3} \\
&\ll x^{1/2}.
\end{aligned}$$

Since $\Psi(x) \ge \Theta(x)$, we get $\Theta(x) = \Psi(x) + O(x^{1/2})$. The Corollary now follows from the improved estimate for Psi.

Proof of Improved Prime Number Theorem: We have

$$\Theta(n) - \Theta(n-1) = \begin{cases} \log n & \text{if } n \text{ is a prime,} \\ 0 & \text{otherwise.} \end{cases}$$

Using this and the Corollary above,

$$\begin{aligned}
\pi(x) &= \sum_{\substack{p\le x \\ p \text{ prime}}} 1 \\
&= \sum_{2\le n\le x} \frac{1}{\log n}\big(\Theta(n) - \Theta(n-1)\big) \\
&= \sum_{2\le n\le x} \frac{1}{\log n} + \sum_{2\le n\le x} \frac{1}{\log n}\Big(\big(\Theta(n) - n\big) - \big(\Theta(n-1) - (n-1)\big)\Big) \\
&= \sum_{2\le n\le x} \frac{1}{\log n} + \left(\frac{\Theta([x]) - [x]}{\log[x]} - \frac{\Theta(1) - 1}{\log 2}\right) + \sum_{2\le n\le x-1} \big(\Theta(n) - n\big)\left(\frac{1}{\log n} - \frac{1}{\log(n+1)}\right) \qquad (1) \\
&= \sum_{2\le n\le x} \frac{1}{\log n} + O\left(\frac{x}{\exp(c\sqrt{\log x})}\right) + O\left(\sum_{2\le n\le x} \frac{n}{\exp(c\sqrt{\log n})} \int_n^{n+1} \frac{1}{u(\log u)^2}\, du\right) \\
&= \sum_{2\le n\le x} \frac{1}{\log n} + O\left(\frac{x}{\exp(c\sqrt{\log x})}\right) + O\left(\int_2^x \frac{1}{\exp(c\sqrt{\log u})(\log u)^2}\, du\right).
\end{aligned}$$

Since

$$\frac{1}{\log n} \le \int_{n-1}^{n} \frac{1}{\log u}\, du \le \frac{1}{\log(n-1)}$$

for $n \geq 3$, we see that

$$\int_2^{[x]+1} \frac{1}{\log u}\, du \leq \sum_{2 \leq n \leq x} \frac{1}{\log n} \leq \int_1^{[x]} \frac{1}{\log u}\, du$$

so that

$$\sum_{2 \leq n \leq x} \frac{1}{\log n} = \mathrm{li}(x) + O(1). \tag{2}$$

Since $(1/2)\log u \geq \sqrt{\log u}$ for $u \geq e$, we easily see that

$$\begin{aligned}
\int_2^x \frac{1}{\exp(c\sqrt{\log u})(\log u)^2}\, du &\ll \int_2^x \frac{1}{\exp(c\sqrt{\log u})}\, du \\
&= \int_2^x \frac{u^{1/2}}{\exp(c\sqrt{\log u})u^{1/2}}\, du \\
&\ll \frac{x^{1/2}}{\exp(c\sqrt{\log x})} \int_2^x \frac{1}{u^{1/2}}\, du \\
&\ll \frac{x}{\exp(c\sqrt{\log x})}.
\end{aligned} \tag{3}$$

Combining (1)-(3) yields our result.

Recall that a crucial point in the proof of the prime number theorem was that the zeta function is non-vanishing on the line $\sigma = 1$. It should come as no surprise that our improved estimate for Psi, which leads to the improved prime number theorem, will depend on a stronger non-vanishing result for the zeta function. Indeed, the major part of our proof for the improved estimate of Psi will hinge on the following.

Theorem 1: There is a constant $c > 0$ such that $\zeta(s) \neq 0$ for all $s = \sigma + it$ with

$$\sigma \geq 1 - \frac{c}{\log(4 + |t|)}.$$

One would very much like to extend the zero-free region (of the zeta function) to the left of the line $\sigma = 1$. Indeed, the Riemann Hypothesis is that this is the case for all $\sigma > 1/2$. Such an estimate would improve the error terms above as follows.

Remark: Suppose all zeros ρ of the zeta function satisfy $\Re(\rho) \leq c$ for some $c < 1$. Then for all $x \geq 2$

$$\Psi(x) = x + O\big(x^c(\log x)^2\big),$$

$$\Theta(x) = x + O\big(x^c(\log x)^2\big),$$

$$\pi(x) = \mathrm{li}(x) + O\big(x^c \log x\big).$$

Since the functional equation implies that $1 - \rho$ is a zero any time ρ is a non-trivial zero of the zeta function, the Reimann Hypothesis is a best-possible result. Unfortunately the hypothesis in the remark above is out of reach, thus we are relegated to working with the less desirable zero-free region of Theorem 1.

Our proof of Theorem 1 involves several intermediate steps.

Lemma 1 (Jensen's Inequality): Let $R > 0$ and suppose $F(s)$ is analytic on the closed disk $\{s\colon |s| \le R\}$. Suppose further that $F(0) \ne 0$ and that $|F(s)| \le M$ for all s in this closed disk. Then for all $r < R$, the number of zeros of F on the closed disk $\{s\colon |s| \le r\}$, counted with multiplicity, is no greater than

$$\frac{\log M - \log|F(0)|}{\log R - \log r}.$$

Proof: The set of zeros of F on the closed disk $\{s \in \mathbb{C}\colon |s| \le R\}$ is necessarily finite, since otherwise it would contain an accumulation point, which in turn would imply that F is identically 0. Let $\rho_1, \dots, \rho_k$ denote the zeros of F, written with multiplicity, on the closed disk of radius R. We may assume that $k > 0$, since otherwise we're done. Now we set

$$G(s) = F(s) \prod_{j=1}^{k} \frac{R^2 - s\overline{\rho_j}}{R(s - \rho_j)}.$$

By construction, this function is analytic on the closed disk $\{s \in \mathbb{C}\colon |s| \le R\}$. Further, for any $s = Re^{i\theta}$ on the boundary of this disk and all $j = 1, \dots, k$ we have

$$\frac{|R^2 - s\overline{\rho_j}|}{|R(s - \rho_j)|} = \frac{|R - e^{i\theta}\overline{\rho_j}|}{|Re^{i\theta} - \rho_j|} = \frac{|\overline{R - e^{-i\theta}\rho_j}|}{|R - e^{-i\theta}\rho_j|} = 1.$$

Thus $|G| \le M$ on the boundary of this disk, so that $|G(0)| \le M$ by the maximum modulus principle.

Now suppose there are L zeros ρ_j with $|\rho_j| \le r$, counted with multiplicity. Then since $r < R$,

$$\begin{aligned} M \ge |G(0)| = |F(0)| \prod_{\substack{1 \le j \le k \\ |\rho_j| \le r}} \frac{R}{|\rho_j|} \prod_{\substack{1 \le j \le k \\ |\rho_j| > r}} \frac{R}{|\rho_j|} \\ \ge |F(0)| \left(\frac{R}{r}\right)^L. \end{aligned}$$

Taking logarithms gives the result.

Lemma 2 (Borel-Carathéodory): Suppose $F(s)$ is analytic on the closed disk $\{s\colon |s| \le R\}$ for some $R > 0$, $F(0) = 0$ and $\Re\big(F(s)\big) \le M$ for all s in the closed disk. Then for all real positive $r < R$ and all complex s with $|s| \le r$,

$$|F(s)| \le \frac{2Mr}{R - r}$$

and

$$|F'(s)| \le \frac{2Mr}{(R - r)^2}.$$

Proof: By Cauchy's integral formula,

$$\frac{1}{2\pi} \int_0^{2\pi} F(Re^{i\theta})\, d\theta = \int_0^1 F(Re^{2\pi i\theta})\, d\theta = \frac{1}{2\pi i} \oint F(s) s^{-1}\, ds = F(0) = 0,$$

where the curvilinear integral is the circle $\{s\colon |s|=R\}$ oriented in the positive (counter-clockwise) direction (set $s=Re^{2\pi i\theta}$). More generally (using the same s and integral path), for all integers $l\geq 0$

$$\int_0^1 F(Re^{2\pi i\theta})e^{2\pi il\theta}\,d\theta = \frac{1}{R^l 2\pi i}\oint F(s)s^{l-1}\,ds = 0$$

and

$$\int_0^1 F(Re^{2\pi i\theta})e^{-2\pi il\theta}\,d\theta = \frac{R^l}{2\pi i}\oint F(s)s^{-l-1}\,ds = \frac{R^l F^{(l)}(0)}{l!}.$$

Using this, for any integer $l\geq 0$ and any real ϕ,

$$\begin{aligned}\frac{R^l F^{(l)}(0)e^{-2\pi i\phi}}{l!} &= \int_0^1 F(Re^{2\pi i\theta})\big(2+e^{2\pi i\phi}e^{2\pi il\theta}+e^{-2\pi i\phi}e^{-2\pi il\theta}\big)\,d\theta \\ &= \int_0^1 F(Re^{2\pi i\theta})\big(2+2\Re(e^{2\pi i(\phi+l\theta)})\big)\,d\theta \\ &= \int_0^1 F(Re^{2\pi i\theta})\big(2+2\cos(2\pi(\phi+l\theta))\big)\,d\theta.\end{aligned}$$

Taking real parts of both sides, we get

$$\frac{R^l}{l!}\Re\big(e^{-2\pi i\phi}F^{(l)}(0)\big)\leq 2M\int_0^1 1+\cos\big(2\pi(\phi+l\theta)\big)\,d\theta = 2M$$

for all positive integers l. We choose ϕ such that $e^{-2\pi i\phi}F^{(l)}(0)=|F^{(l)}(0)|$ to see that

$$\frac{F^{(l)}(0)}{l!}\leq\frac{2M}{R^l}\qquad\text{all } l\geq 1. \tag{4}$$

Now since F is analytic in the closed disk and $F(0)=0$, it has a power series representation

$$F(s)=\sum_{l\geq 1}\frac{F^{(l)}(0)s^l}{l!}$$

valid for all s with $|s|<R$. Now assuming that $|s|\leq r<R$ we have by (4)

$$|F(s)|\leq\sum_{l\geq 1}\frac{|F^{(l)}(0)||s|^l}{l!}\leq 2M\sum_{l\geq 1}(r/R)^l=\frac{2Mr}{R-r}.$$

Similarly for the derivative,

$$|F'(s)|\leq\sum_{l\geq 1}\frac{|F^{(l)}(0)||s^{l-1}|}{(l-1)!}\leq\frac{2M}{R}\sum_{l\geq 1}l(r/R)^{l-1}=\frac{2Mr}{(R-r)^2}.$$

Lemma 3: Suppose $F(s)$ is analytic on the closed disk $\{s\colon |s|\leq 1\}$, $|F(s)|\leq M$ on this disk and $F(0)\neq 0$. Then for all r and R with $0<r<R<1$ and all s with $|s|\leq r$,

$$\left|\frac{d\log F(s)}{ds}-\sum_{j=1}^{k}\frac{1}{s-\rho_j}\right| = O\big(\log M-\log|F(0)|\big),$$

where $\rho_1, \dots, \rho_k$ are the zeros of F (with multiplicity) on the closed disk $\{s\colon |s| \le R\}$ and the implicit constant depends on r and R.

Proof: As noted in the proof of Jensen's Inequality, the set of zeros of F on the closed disk $\{s\colon |s| \le 1\}$ is finite. By possibly replacing R with a slightly larger number, we may assume that $F(s) \neq 0$ for all s with $|s| = R$. Applying Jensen's Inequality (with R there equal to 1 here and r there equal to R here), we see that

$$k \le \frac{\log M - \log |F(0)|}{-\log R} \ll \log M - \log |F(0)|. \tag{5}$$

Letting

$$G(s) = F(s) \prod_{j=1}^{k} \frac{R^2 - s\overline{\rho_j}}{R(s-\rho_j)}$$

as in the proof of Jensen's Inequality, we see that G is analytic on the closed disk $\{s\colon |s| \le R\}$, $G(s) \neq 0$ on this disk,

$$|G(0)| = |F(0)| \prod_{j=1}^{k} \frac{R}{|\rho_j|} \ge |F(0)|,$$

and $|G(s)| \le M$ on this disk.

Now the function $H(s) = \log G(s) - \log G(0)$ is analytic on the closed disk of radius R, $H(0) = 0$ and

$$\Re(H(s)) = \log |G(s)| - \log |G(0)| \le \log M - \log |F(0)|.$$

We apply the Borel-Carathéodory lemma to the function $H(s)$, getting

$$|H'(s)| \le \frac{2(\log M - \log |F(0)|)r}{(R-r)^2} \ll \log M - \log |F(0)| \tag{6}$$

for all s with $|s| \le r$. On the other hand, we have

$$\begin{aligned} H'(s) = \frac{d \log G(s)}{ds} &= \frac{d \log F(s)}{ds} + \sum_{j=1}^{k} \frac{d \log(R^2 - s\overline{\rho_j})}{ds} - \sum_{j=1}^{k} \frac{d \log R + \log(s-\rho_j))}{ds} \\ &= \frac{d \log F(s)}{ds} + \sum_{j=1}^{k} \frac{1}{s - (R^2/\overline{\rho_j})} - \sum_{j=1}^{k} \frac{1}{s-\rho_j}. \end{aligned}$$

In addition, if $|s| \le r$ we have $|s - (R^2/\overline{\rho_j})| \ge R - r$ for all $j = 1, \dots, k$, so that by (5) and (6)

$$\begin{aligned} \left| \frac{d \log F(s)}{ds} - \sum_{j=1}^{k} \frac{1}{s-\rho_j} \right| &= \left| H'(s) - \sum_{j=1}^{k} \frac{1}{s - (R^2/\overline{\rho_j})} \right| \\ &\le |H'(s)| + \sum_{j=1}^{k} \frac{1}{|s - (R^2/\overline{\rho_j})|} \\ &\ll \log M - \log |F(0)| + \frac{k}{R-r} \\ &\ll \log M - \log |F(0)|. \end{aligned}$$

Lemma 4: For all $s = \sigma + it$ with $5/6 \le \sigma \le 13/6$ and $|t| \ge 7/8$,

$$\left| \frac{d \log \zeta(s)}{ds} - \sum_{\rho} \frac{1}{s-\rho} \right| = O\big(\log(|t|+4) \big),$$

where the sum is over all zeros ρ of ζ satisfying $|\rho - (3/2+it)| \le 5/6$ with multiplicity, and the implicit constant is absolute.

Proof: Fix a $t_0 \in \mathbb{R}$ with $|t_0| \ge 7/8$. We will apply Lemma 3 to the function $F(s') = \zeta(s' + 3/2 + it_0)$, with $R = 5/6$ and $r = 2/3$. Since the only pole of $\zeta(s)$ is $s = 1$, the only pole of $F(s')$ is $s' = -1/2 - it_0$, which is not in the closed disk of radius 1 since $(1/2)^2 + |t_0|^2 \ge 65/64$. Thus $F(s')$ is analytic on the closed disk $\{s' \colon |s'| \le 1\}$. We also have

$$F(0) = \zeta\big(3/2 + it_0\big) = \prod_{p \text{ prime}} \big(1 - p^{-3/2 - it_0}\big)^{-1} \neq 0$$

by the Euler product (valid here since the real part of the input is greater than 1). Thus by Lemma 3

$$\left| \frac{d \log F(s')}{ds'} - \sum_{j=1}^{k} \frac{1}{s' - \rho_j'} \right| = O\big(\log M - \log |F(0)| \big), \tag{7}$$

where $\rho_1', \dots, \rho_k'$ are the zeros of $F(s')$ (with multiplicity) on the closed disk $\{s' \colon |s'| \le 5/6\}$ and M is the supremum of $|F(s')|$ on the disk $\{s' \colon |s'| \le 1\}$.

We note that any $s = \sigma + it$ with $5/6 \le \sigma \le 13/6$ and $t = t_0$ can be written as a sum $s = s' + 3/2 + it_0$ for some s' with $|s'| \le 2/3$. Any zero ρ_j' of $F(s')$ is of the form $\rho_j' = \rho_j - 3/2 - it$, where ρ_j is a zero of $\zeta(s)$ and $s' - \rho_j' = s - \rho_j$. Since ρ_j' is in the closed disk of radius $R = 5/6$, $|\rho_j - (3/2+it)| \le 5/6$. We also have

$$\frac{d \log F(s')}{ds'} = \frac{d \log \zeta(s)}{ds'} = \frac{d \log \zeta(s)}{ds}.$$

Therefore the lemma follows from (7) once we get an appropriate upper bound for $|F(s')|$ on $\{s' \colon |s'| \le 1\}$ and lower bound for $|F(0)|$.

We have $\sigma = \Re(s') + 3/2 \ge -1 + 3/2 = 1/2$ for all s' with $|s'| \le 1$. In particular, we may use the representation

$$F(s') = \zeta(s) = \frac{1}{1 - 2^{1-s}} \sum_{n \ge 1} \frac{(-1)^{n+1}}{n^s}. \tag{8}$$

We saw above that $|s-1| \ge \sqrt{65/64}$. This implies that $|1 - 2^{1-s}| \gg 1$, so that

$$\left| \frac{1}{1 - 2^{1-s}} \right| \ll 1. \tag{9}$$

Next, the Dirichlet series $\sum_{n\geq 1}\frac{(-1)^{n+1}}{n^s}$ has abscissa of convergence 0. Since $\Re(s)\geq 1/2>0$, by Theorem 2 from the Dirichlet Series handout we have

$$\sum_{n\geq 1}\frac{(-1)^{n+1}}{n^s}=s\int_1^\infty\sum_{1\leq n\leq x}(-1)^{n+1}x^{-(s+1)}\,dx.$$

Hence

$$\begin{aligned}\left|\sum_{n\geq 1}\frac{(-1)^{n+1}}{n^s}\right| &= |s|\left|\int_1^\infty\sum_{1\leq n\leq x}(-1)^{n+1}x^{-(s+1)}\,dx\right|\\ &\leq |s|\int_1^\infty x^{-(\sigma+1)}\,dx\\ &\leq |s|\int_1^\infty x^{-3/2}\,dx\\ &= 2|s|\\ &= 2\sqrt{\sigma^2+t^2}\\ &\ll 4+|t|.\end{aligned} \tag{10}$$

Via (8)-(10), we see that we may use $\log(4+|t|)$ in place of $\log M$ in (7). All that remains is to bound $|F(0)|$ away from 0. But that is relatively simple. A rather crude estimate gives $|1-n^{-(3/2+it)}|\leq 1+n^{-3/2}$ for all $n\geq 1$. Thus by exercise 11

$$|F(0)|=\prod_{p\text{ prime}}|1-p^{-(3/2+it)}|^{-1}\geq\prod_{p\text{ prime}}\left(1+p^{-3/2}\right)^{-1}=\frac{\zeta(2\cdot 3/2)}{\zeta(3/2)}.$$

Using this estimate in (7) completes the proof.

Lemma 5: For all $\sigma>1$ and all t

$$-3\Re\left(\frac{\zeta'(\sigma)}{\zeta(\sigma)}\right)-4\Re\left(\frac{\zeta'(\sigma+it)}{\zeta(\sigma+it)}\right)-\Re\left(\frac{\zeta'(\sigma+i2t)}{\zeta(\sigma+i2t)}\right)\geq 0.$$

Proof: Recall from our proof of the theorem of Hademard and de la Vallée Poussin that

$$\log\zeta(s)=\sum_{n\geq 2}\frac{a_n}{n^s},$$

where

$$a_n=\begin{cases}\frac{1}{k} & \text{if } n=p^k \text{ for some prime } p,\\ 0 & \text{otherwise.}\end{cases}$$

In other words,

$$\log\zeta(s)=\sum_{n\geq 2}\frac{\Lambda(n)}{(\log n)n^s}.$$

This is absolutely convergent when $\Re(s)>1$, so by results from the handout on Dirichlet series we may differentiate this term-by-term to get

$$\frac{-\zeta'(s)}{\zeta(s)}=-\frac{d\log\zeta(s)}{ds}=\sum_{n\geq 1}\frac{\Lambda(n)}{n^s}.$$

Using this, we get

$$
\begin{aligned}
-3\Re\left(\frac{\zeta'(\sigma)}{\zeta(\sigma)}\right) - 4\Re\left(\frac{\zeta'(\sigma+it)}{\zeta(\sigma+it)}\right) - \Re\left(\frac{\zeta'(\sigma+i2t)}{\zeta(\sigma+i2t)}\right) &= \sum_{n\geq 2}\frac{\Lambda(n)}{n^\sigma}\big(3+4\cos(t\log n)+\cos(2t\log n)\big) \\
&= \sum_{n\geq 2}\frac{\Lambda(n)}{n^\sigma}\big(2(1+\cos(t\log n))\big)^2 \\
&\geq 0.
\end{aligned}
$$

Proof of Theorem 1: We already proved that $\zeta(s)\neq 0$ whenever $\sigma = 1$ in our proof of the prime number theorem. By the Euler product, $\zeta(s)\neq 0$ whenever $\sigma > 1$. Therefore it suffices to only consider possible zeros with real part less than 1.

We first consider the simplest case of zeros ρ with $|\Im(\rho)| \leq 7/8$. The function

$$
F(s) := \begin{cases} (s-1)\zeta(s) & \text{if } s\neq 1, \\ 1 & \text{if } s = 1 \end{cases}
$$

is analytic, thus uniformly continuous on the compact set

$$
S = \{s = \sigma + it \colon 5/6 \leq \sigma \leq 1,\ |t| \leq 7/8\}.
$$

Set $m = \inf_{|t|\leq 7/8}\{|F(1+it)|\} > 0$; by the uniform continuity above there is an $\epsilon_0 > 0$ such that $|F(s_1) - F(s_2)| < m$ whenever $s_1, s_2 \in S$ with $|s_1 - s_2| < \epsilon_0$. In particular, letting $\Re(s_2) = 1$ here gives

$$
\zeta(\sigma+it)\neq 0 \qquad \text{all } \sigma \geq 1-\epsilon/2, \quad |t|\leq 7/8. \tag{11}
$$

Suppose now that $\rho_0 = \sigma_0 + it_0$ is a zero of ζ with $5/6 \leq \sigma_0 < 1$ and $|t_0| \geq 7/8$ (if there are no such zeros, we're already done by (11)). Let $1 \geq \delta > 0$ to be determined later and set $s = 1 + \delta + it_0$. We apply Lemma 4 and get

$$
\left|\frac{\zeta'(s)}{\zeta(s)} - \sum_\rho \frac{1}{s-\rho}\right| \leq c_1(\log|t_0| + 4)
$$

for some constant $c_1 \geq 1$ (independent of s and ρ_0), where the sum is over zeros ρ of ζ with $|\rho - (3/2 + it_0)| \leq 5/6$. We note that $\Re(1/(s-\rho)) = \Re(s-\rho)/|s-\rho|^2 > 0$ (since $1/(1+\delta+\Re(\rho)) > 0$) for all of these zeros, one of which is ρ_0. Thus the above inequality easily implies that

$$
\begin{aligned}
-\Re\left(\frac{\zeta'(1+\delta+it_0)}{\zeta(1+\delta+it_0)}\right) &\leq -\Re\left(\sum_\rho \frac{1}{s-\rho}\right) + c_1\log(|t_0|+4) \\
&\leq \frac{-1}{1+\delta-\sigma_0} + c_1\log(|t_0|+4).
\end{aligned} \tag{12}
$$

The same argument with $s = 1 + \delta + i2t_0$ gives

$$
-\Re\left(\frac{\zeta'(1+\delta+i2t_0)}{\zeta(1+\delta+i2t_0)}\right) \leq c_1\log(2|t_0|+4). \tag{13}
$$

We previously saw that both $\zeta(s) - s^{-1}$ and $\zeta'(s) + s^{-2}$ are analytic in the entire complex. Thus there is a positive constant c_2 such that

$$\max\left\{|\zeta'(1+\delta)+\delta^{-2}|,\ |\zeta(1+\delta)-\delta^{-1}|\right\} \le c_2 \tag{14}$$

for all positive $\delta \le 1$. We also have $\zeta(1+\delta) \ge \zeta(2) > 1$. Since $\lim_{\sigma\to 1^+}(\sigma-1)\zeta(\sigma) = 1$, there is a positive constant c_3 such that $\delta\zeta(1+\delta) \le c_3$ for all positive $\delta \le 1$. Using these inequalities together with (14) yields

$$\begin{aligned}
-\Re\left(\frac{\zeta'(1+\delta)}{\zeta(1+\delta)}\right) - \delta^{-1} &\le \left|\frac{-\zeta'(1+\delta)}{\zeta(1+\delta)} - \frac{1}{\delta}\right| \\
&= \left|\frac{-\zeta'(1+\delta)\delta^{-1} - \zeta(1+\delta)\delta^{-2}}{\zeta(1+\delta)\delta^{-1}}\right| \\
&= \frac{\left|\big(-\zeta'(1+\delta)-\delta^{-2}\big)\delta^{-1} + \delta^{-2}\big(\delta^{-1}-\zeta(1+\delta)\big)\right|}{\zeta(1+\delta)\delta^{-1}} \\
&\le \frac{|\zeta'(1+\delta)+\delta^{-2}|}{\zeta(1+\delta)} + \frac{|\zeta(1+\delta)-\delta^{-1}|}{\delta\zeta(1+\delta)} \\
&\le c_2 + c_2/c_3.
\end{aligned} \tag{15}$$

We now combine (12), (13) and (15) to get

$$\begin{aligned}
&-3\Re\left(\frac{\zeta'(1+\delta)}{\zeta(1+\delta)}\right) - 4\Re\left(\frac{\zeta'(1+\delta+it_0)}{\zeta(1+\delta+it_0)}\right) - \Re\left(\frac{\zeta'(1+\delta+i2t_0)}{\zeta(1+\delta+i2t_0)}\right) \\
&\qquad\le \frac{3}{\delta} - \frac{4}{1+\delta-\sigma_0} + 3c_2 + 3c_2/c_3 + 4c_1\log(|t_0|+4) + c_1\log(2|t_0|+4).
\end{aligned}$$

Since $c_1 \ge 1$ and $\log(|t_0|+4) > \log 4 > 1$, we may set $\delta^{-1} = c_1\log(|t_0|+4)$. With this choice of δ, the above inequality together with Lemma 5 gives

$$\frac{4}{1+c_1\log(|t_0|+4)-\sigma_0} \le 3\log(|t_0|+4) + 3c_2 + 3c_2/c_3 + 4c_1\log(|t_0|+4) + c_1\log(2|t_0|+4),$$

clearly implying that

$$1-\sigma_0 \ge \frac{c}{\log(|t_0|+4)}$$

for some positive constant c. This together with (11) completes the proof.

Theorem 2: Let c be the constant in the statement of Theorem 1 and set $c' = \min\{c/2, 1/6\}$. For all $s = \sigma+it$ with $\sigma > 1 - c'/\log(|t|+4)$ and $|t| \ge 7/8$ we have

$$\left|\frac{\zeta'(s)}{\zeta(s)}\right| \ll \log(|t|+4),$$

where the implicit constant is absolute. If $\sigma > 1 - c'/\log(|t|+4)$ and $|t| < 7/8$, then

$$-\frac{\zeta'(s)}{\zeta(s)} = \frac{1}{s-1} + O(1).$$

Proof: We note that the c in Theorem 1 is no greater than the ϵ in (11). Thus $\zeta'(s)/\zeta(s)$ is analytic for all $s \neq 1$ with $\sigma \geq 1 - c/2$ and $|t| \leq 7/8$. We've seen that $\zeta'(s)/\zeta(s)$ has a simple pole of residue -1 at $s = 1$. Thus there is an M such that

$$\left|\frac{\zeta'(s)}{\zeta(s)} + \frac{1}{s-1}\right| \leq M$$

for all s with $|t| \leq 7/8$ and $1 - c/2 \leq \sigma \leq 2$, say. Therefore we may assume from now on that either $|t| > 7/8$ or $\sigma > 2$.

Suppose that $\sigma > 1$. Then as in the proof of Lemma 5

$$\begin{aligned}\left|\frac{\zeta'(s)}{\zeta(s)}\right| &= \left|\sum_{n\geq 2} \frac{\Lambda(n)}{n^s}\right| \\ &\leq \sum_{n\geq 2} \frac{\Lambda(n)}{n^\sigma} \\ &= -\frac{\zeta'(\sigma)}{\zeta(\sigma)}.\end{aligned}$$

Now using (15) gives

$$\left|\frac{\zeta'(\sigma)}{\zeta(\sigma)}\right| \ll \frac{1}{\sigma - 1}.$$

In particular, the theorem is true whenever $\sigma \geq 1 + 1/\log(|t| + 4)$ (which clearly includes the case where $\sigma > 2$).

It remains to deal with the case where $|t| > 7/8$ and $\sigma < 1 + 1/\log(|t| + 4)$. Suppose $s = \sigma + it$ with $1 - c'/\log(|t| + 4) < \sigma < 1 + 1/\log(|t| + 4)$ and $|t| > 7/8$ and set $s_0 = \sigma_0 + it_0$ with $t_0 = t$ and $\sigma_0 = 1 + 1/\log(|t_0| + 4)$. Since $c'/\log(|t| + 4) < 1/6$ we may apply Lemma 4 to both s and s_0; we get

$$\begin{aligned}\left|\frac{-\zeta'(s)}{\zeta(s)} - \sum_\rho \frac{1}{s-\rho}\right| &= O\big(\log(|t| + 4)\big) \\ \left|\frac{-\zeta'(s_0)}{\zeta(s_0)} - \sum_\rho \frac{1}{s_0-\rho}\right| &= O\big(\log(|t| + 4)\big),\end{aligned} \tag{16}$$

where the sums are over all zeros ρ of ζ satisfying $|\rho - (3/2 + it)| \leq 5/6$, written with multiplicity. Since the theorem holds for s_0, we get

$$\left|\frac{\zeta'(s_0)}{\zeta(s_0)}\right|, \quad \left|\sum_\rho \frac{1}{s_0-\rho}\right| \ll \log(|t| + 4). \tag{17}$$

By construction we have $|s - s_0| \leq (1 + c/2)/\log(|t| + 4)$. By Theorem 1 we must have $|s - \rho| \geq c/2 \log(|t| + 4)$ for all ρ, and since all ρ have $\Re(\rho) < 1$ we must have $|s_0 - \rho| \geq 1/\log(|t| + 4)$. Thus for all ρ

$$\begin{aligned}\left|\frac{s-\rho}{s_0-\rho}\right| &= \left|\frac{s_0 - \rho + s - s_0}{s_0 - \rho}\right| \\ &\leq 1 + \frac{|s - s_0|}{|s_0 - \rho|} \\ &\ll 1\end{aligned}$$

and

$$\begin{aligned}\left|\frac{s_0-\rho}{s-\rho}\right| &= \left|\frac{s-\rho+s_0-s}{s-\rho}\right| \\ &\le 1+\frac{|s-s_0|}{|s-\rho|} \\ &\ll 1.\end{aligned}$$

Therefore

$$\begin{aligned}\sum_\rho\left|\frac{1}{s-\rho}-\frac{1}{s_0-\rho}\right| &= \sum_\rho\frac{|s-s_0|}{|s-\rho|\cdot|s_0-\rho|} \\ &\ll \sum_\rho\frac{1/\log(|t|+4)}{|s_0-\rho|^2} \\ &\le \sum_\rho\frac{\Re(s_0-\rho)}{|s_0-\rho|^2} \\ &= \sum_\rho\Re\left(\frac{1}{s_0-\rho}\right) \\ &\ll \log(|t|+4)\end{aligned}$$

by (17). Combining this with (16) and (17) completes the proof.