Math 680 Fall 2017

A Quantitative Prime Number Theorem II:

The Inverse Mellin Transform

Recall that our main goal is a proof of

Theorem (Improved Estimate for Psi): There is a positive $c < 1$ such that

$$\Psi(x) = x + O\left(\frac{x}{\exp(c\sqrt{\log x})}\right).$$

Given our zero-free region estimates from the first part, this will follow from the following.

Theorem 3: Suppose $D(s) = \sum_{n\geq 1} a_n n^{-s}$ is a Dirichlet series with finite abscissa of convergence and $T \geq 1$. Then for all $\sigma > \max\{0, \sigma_a\}$ and positive x,

$$\left| \sideset{}{'}\sum_{1\leq n\leq x} a_n - \frac{1}{2\pi i}\int_{\sigma-iT}^{\sigma+iT} D(s)\frac{x^s}{s}\,ds \right| \ll \sum_{\substack{x/2<n<2x \\ n\neq x}} |a_n| \min\{1, x/T|x-n|\} + \frac{(4x)^\sigma}{T}\sum_{n\geq 1}\frac{|a_n|}{n^\sigma},$$

where

$$\sideset{}{'}\sum_{1\leq n\leq x} a_n = \begin{cases} \sum_{1\leq n<x} a_n & \text{if } x \notin \mathbb{Z}, \\ \sum_{1\leq n<x} a_n + a_x/2 & \text{if } x \in \mathbb{Z} \end{cases}$$

and the implicit constant is absolute.

We'll prove Theorem 3 in due course, but first see how it applies to prove our improved estimate for $\Psi(x)$. We apply Theorem 3 to the case where $a_n = \Lambda(n)$, getting

$$\begin{aligned} \Psi(x) &= \sum_{1\leq n\leq x} \Lambda(n) \\ &= \frac{-1}{2\pi i}\int_{\sigma-iT}^{\sigma+iT} \frac{\zeta'(s)}{\zeta(s)}\frac{x^s}{s}\,ds + O\left(\log x + \sum_{\substack{x/2<n<2x \\ n\neq x}} \Lambda(n)\min\{1, x/T|x-n|\} + \frac{(4x)^\sigma}{T}\sum_{n\geq 1}\frac{\Lambda(n)}{n^\sigma}\right) \end{aligned} \tag{18}$$

for all $\sigma > 1$. By (15)

$$\sum_{n\geq 1}\frac{\Lambda(n)}{n^\sigma} = -\frac{\zeta'(\sigma)}{\zeta(\sigma)} \gg\ll \frac{1}{\sigma - 1}.$$

Using this estimate, we get

$$
\begin{aligned}
&\log x + \sum_{\substack{x/2<n<2x\\ n\neq x}} \Lambda(n)\min\{1, x/T|x-n|\} + \frac{(4x)^\sigma}{T}\sum_{n\geq 1}\frac{\Lambda(n)}{n^\sigma}\\
&= O\left(\log x\left[1+\frac{x}{T}\sum_{\substack{x/2<n<2x\\ |n-x|\geq 1/2}}\frac{1}{|x-n|}\right]+\frac{(4x)^\sigma}{T(\sigma-1)}\right)\\
&= O\left(\log x\left[1+\frac{x}{T}\sum_{1\leq k\leq x}\frac{1}{k}\right]+\frac{(4x)^\sigma}{T(\sigma-1)}\right)\\
&= O\left(\log x\left[1+\frac{x\log x}{T}\right]+\frac{(4x)^\sigma}{T(\sigma-1)}\right).
\end{aligned} \tag{19}
$$

It remains to estimate the integral in (18). To do that, we will use Theorems 1 and 2. Set $\sigma' = 1-c/\log T$ for a suitably small positive c. Consider the rectangular contour C with vertices $\sigma' \pm iT$ and $\sigma \pm iT$. Then by Theorem 1 $\zeta'(s)/\zeta(s)$ has only the simple pole of residue -1 at $s = 1$ inside this contour. Thus by Cauchy's theorem

$$
\frac{-1}{2\pi i}\oint_C \frac{\zeta'(s)}{\zeta(s)}\frac{x^s}{s}\,ds = x. \tag{20}
$$

Further, by Theorem 2 $|\zeta'(s)/\zeta(s)| \ll \log T$ along the top and bottom of the contour (we're assuming that T is "large"). We easily see that $|x^s/s| \ll x^\sigma/T$ along the top and bottom of the contour, so that

$$
\left|\int_{\sigma'\pm iT}^{\sigma\pm iT}\frac{\zeta'(s)}{\zeta(s)}\frac{x^s}{s}\,ds\right| = O\left(\frac{\log T x^\sigma(\sigma-\sigma')}{T}\right). \tag{21}
$$

Finally, for the left side of the contour we have by Theorem 2 once more (with $s = \sigma' + it$)

$$
\begin{aligned}
\left|\int_{\sigma'-iT}^{\sigma'+iT}\frac{\zeta'(s)}{\zeta(s)}\frac{x^s}{s}\,ds\right| &\leq \left|\int_{1\leq|t|\leq T}\frac{\zeta'(s)}{\zeta(s)}\frac{x^s}{s}\,dt\right| + \left|\int_{1\geq|t|}\frac{\zeta'(s)}{\zeta(s)}\frac{x^s}{s}\,dt\right|\\
&\ll x^{\sigma'}\log T\int_{1\leq|t|\leq T}\frac{1}{1+|t|}\,dt + x^{\sigma'}\int_{-1}^{1}\frac{1}{|\sigma'+it-1|}\,dt\\
&\ll x^{\sigma'}\log T\int_1^T\frac{1}{t}\,dt + \frac{1}{1-\sigma'}\int_{-1}^{1}\,dt\\
&\ll x^{\sigma'}(\log T)^2.
\end{aligned} \tag{22}
$$

Exercise 20: Assume $x \geq T \geq 2$. When $\sigma = 1 + 1/\log x$, show that

$$
\log x\left[1+\frac{x\log x}{T}\right]+\frac{(4x)^\sigma}{T(\sigma-1)} \ll \frac{x(\log x)^2}{T}
$$

and

$$
\frac{\log T x^\sigma(\sigma-\sigma')}{T} \ll \frac{x}{T}
$$

for all $0 < \sigma' < 1$.

Exercise 21: Show that $x^{c/\log T} = T$ when $T = \exp(\sqrt{c \log x})$, and for this value of T

$$x^{\sigma'}(\log T)^2 + \frac{x(\log x)^2}{T} \ll x(\log x)^2\left(x^{-c/\log T} + T^{-1}\right)$$
$$\ll \frac{x}{\exp(c\sqrt{\log x})}.$$

Conclude that

$$\Psi(x) = x + O\left(\frac{x}{\exp(c\sqrt{\log x})}\right).$$

Exercise 22: Show that the limit of curvilinear integrals

$$\lim_{T\to\infty} \frac{1}{2\pi i}\int_{\sigma_0 - iT}^{\sigma_0+iT} \frac{y^s}{s}\, ds = \begin{cases} 1 & \text{if } y > 1, \\ 1/2 & \text{if } y = 1, \\ 0 & \text{if } 0 < y < 1 \end{cases}$$

for all $\sigma_0 > 0$. In fact,

$$\frac{1}{2\pi i}\int_{\sigma_0 - iT}^{\sigma_0+iT} \frac{y^s}{s}\, ds = \begin{cases} 1 + O(y^{\sigma_0}/T) & \text{if } y \geq 2, \\ O(y^{\sigma_0}/T) & \text{if } 0 < y \leq 1/2. \end{cases}$$

Hint: For the case $y \geq 2$ use Cauchy's theorem on rectangles containing the origin that stretch further and further to the left but remain fixed at real part equal to σ_0 on the right.

Proposition: Suppose $1/2 < y < 2$. Then

$$\frac{1}{2\pi i}\int_{\sigma_0 - iT}^{\sigma_0+iT} \frac{y^s}{s}\, ds = \frac{1}{2} + \frac{1}{\pi}\int_0^{T\log y} \frac{\sin t}{t}\, dt + O(\sigma_0 2^{\sigma_0}/T)$$

for all $\sigma_0 > 0$, where the implicit constant is absolute.

Proof: Let $\epsilon > 0$ and consider the curvilinear integral $\frac{1}{2\pi i}\oint_{C(\epsilon)} \frac{y^s}{s}\, ds$, where the contour $C(\epsilon)$ is the rectangle with vertices $\sigma_0 \pm iT$ and $\pm iT$ except modified along the imaginary axis with the right hand semi-circle centered at the origin of radius ϵ. The semicircular portion ensures that the pole at the origin is avoided, so that this curvilinear integral is zero by Cauchy's (or Green's) theorem. The integral in the statement of the proposition is just the integral along the right hand side of the rectangle. The integrals along the top and bottom portions are clearly $O(\sigma_0 2^{\sigma_0}/T)$.

For the left hand side we have (with $s = it$)

$$\frac{1}{2\pi i}\int_{iT}^{i\epsilon} \frac{y^s}{s}\, ds = \frac{1}{2\pi i}\int_T^{\epsilon} \frac{y^{it}}{it}\, idt$$
$$= \frac{1}{2\pi i}\int_T^{\epsilon} \frac{\cos(t\log y)}{t} + \frac{i\sin(t\log y)}{t}\, dt$$
$$= \frac{1}{2\pi i}\int_{\epsilon}^{T} \frac{-\cos(t\log y)}{t} - \frac{i\sin(t\log y)}{t}\, dt$$

and similarly (with $s = -it$)

$$\frac{1}{2\pi i}\int_{-i\epsilon}^{-iT} \frac{y^s}{s}\,ds = \frac{1}{2\pi i}\int_{\epsilon}^{T} \frac{\cos(t\log y)}{t} - \frac{i\sin(t\log y)}{t}\,dt.$$

For the circular portion of the left hand side, we set $s = \epsilon e^{i\theta}$ and see that this portion is

$$\frac{1}{2\pi i}\int_{\pi/2}^{-\pi/2} \frac{y^{\epsilon e^{i\theta}}}{\epsilon e^{i\theta}} i\epsilon e^{i\theta}\,d\theta = \frac{1}{2\pi}\int_{\pi/2}^{-\pi/2} y^{\epsilon\cos\theta} y^{i\epsilon\sin\theta}\,d\theta.$$

All together then, the left side of the contour contributes

$$\frac{-1}{\pi}\int_{\epsilon}^{T} \frac{\sin(t\log y)}{t}\,dt - \frac{1}{2\pi}\int_{-\pi/2}^{\pi/2} y^{\epsilon\cos\theta} y^{i\epsilon\sin\theta}\,d\theta = \frac{-1}{\pi}\int_{\epsilon\log y}^{T\log y} \frac{\sin x}{x}\,dx - \frac{1}{2\pi}\int_{-\pi/2}^{\pi/2} y^{\epsilon\cos\theta} y^{i\epsilon\sin\theta}\,d\theta.$$

Since the last integral here is clearly absolutely convergent, we are justified in taking the limit as $\epsilon \to 0^+$. The proposition follows.

Exercise 23: For all $s = \sigma + it$ with $0 < \sigma < 1$, show that

$$\int_0^\infty u^{s-1} e^{-iu}\,du = e^{-i\pi s/2}\Gamma(s).$$

Hint: integrate $u^{s-1}e^{-u}$ along the quarter circle contour of radius R in the first quadrant and let $R \to \infty$.

Exercise 24: Using exercise 23, show that

$$\int_0^\infty u^{s-1}\sin u\,du = \Gamma(s)\sin(\pi s/2)$$

for all $s = \sigma + it$ with $0 < \sigma < 1$. Let $\sigma \to 0^+$ to find $\displaystyle\int_0^\infty (\sin u/u)\,du$.

Exercise 25: Show that

$$\left|\int_B^\infty \frac{\sin u}{u}\,du\right| \ll \min\{1, 1/B\}$$

for all $B > 0$. Hint: integrate by parts for the case $B \geq 1$.

Proof of Theorem 3: Since $\sigma > \sigma_a$, the Dirichlet series is absolutely convergent so we may interchange the integral with the summation to get

$$\frac{1}{2\pi i}\int_{\sigma-iT}^{\sigma+iT} D(s)\frac{x^s}{s}\,ds = \sum_{n\geq 1} a_n \frac{1}{2\pi i}\int_{\sigma-iT}^{\sigma+iT} \frac{(x/n)^s}{s}\,ds. \tag{23}$$

By exercise 22 and the Proposition

$$\begin{aligned}\sum_{n\geq 1} a_n \frac{1}{2\pi i}\int_{\sigma-iT}^{\sigma+iT} \frac{(x/n)^s}{s}\,ds &= \sum_{n\leq x/2} a_n\big(1 + O((x/n)^\sigma/T)\big) + \sum_{n\geq 2x} a_n O\big((x/n)^\sigma/T\big) \\ &+ \sum_{x/2<n<2x} a_n\left(\frac{1}{2} + \frac{1}{\pi}\int_0^{T\log(x/n)} \frac{\sin t}{t}\,dt + O(\sigma 2^\sigma/T)\right).\end{aligned} \tag{24}$$

If $x > n > x/2$, exercises 24 and 25 give

$$\begin{aligned} \frac{1}{2} + \frac{1}{\pi}\int_0^{T\log(x/n)} \frac{\sin t}{t}\,dt &= \frac{1}{2} + \frac{1}{\pi}\left(\int_0^\infty \frac{\sin t}{t}\,dt - \int_{T\log(x/n)}^\infty \frac{\sin t}{t}\,dt\right) \\ &= 1 - \frac{1}{\pi}\int_{T\log(x/n)}^\infty \frac{\sin t}{t}\,dt \\ &= 1 + O\big(\min\{1,\ 1/T\log(x/n)\}\big), \end{aligned} \tag{25}$$

and if $x < n < 2x$

$$\begin{aligned} \frac{1}{2} + \frac{1}{\pi}\int_0^{T\log(x/n)} \frac{\sin t}{t}\,dt &= \frac{1}{2} - \frac{1}{\pi}\int_0^{T\log(n/x)} \frac{\sin t}{t}\,dt \\ &= \frac{1}{2} - \frac{1}{\pi}\left(\int_0^\infty \frac{\sin t}{t}\,dt - \int_{T\log(n/x)}^\infty \frac{\sin t}{t}\,dt\right) \\ &= \frac{1}{\pi}\int_{T\log(n/x)}^\infty \frac{\sin t}{t}\,dt \\ &= O\big(\min\{1,\ 1/T\log(n/x)\}\big). \end{aligned} \tag{26}$$

Finally, since $\frac{n}{x} = 1 - \frac{n-x}{x}$, we see that

$$|\log(x/n)| = O\big(|x-n|/x\big) \qquad x/2 < n < 2x. \tag{27}$$

The theorem now follows from (23)-(27).