We saw earlier the xi function \( \xi(s) = \frac{1}{2}s(s-1)\zeta(s)\Gamma(s/2)\pi^{-s/2} \). Our goal is to find an infinite product representation of this function. With that, we’ll get some nice corollaries.

**Lemma 1:** Suppose \( x \) is a positive integer and \( s = \sigma + it \neq 1 \) with \( \sigma > 0 \). Then

\[
\zeta(s) = \sum_{n \leq x} \frac{1}{n^s} + \frac{x^{1-s}}{s-1} - s \int_x^\infty (u-[u])u^{-(s+1)} \, du.
\]

Proof: First assume that \( \sigma > 1 \) and write

\[
\zeta(s) = \sum_{n \leq x} \frac{1}{n^s} + \sum_{n > x} \frac{1}{n^s}.
\]

Now

\[
-x \sum_{n > x} \frac{1}{n^s} = \sum_{n \geq x} -(n+1)^{-s} = \sum_{n \geq x} (n+1)(n+1)^{-s} - (n+1)^{-s} - (n+1)^{1-s} = \sum_{n \geq x} n(n+1)^{-s} - (n+1)^{1-s} = x^{1-s} + \sum_{n \geq x} n(n+1)^{-s} - n^{1-s} = x^{1-s} + \sum_{n \geq x} n((n+1)^{-s} - n^{-s}) = x^{1-s} - \sum_{n \geq x} ns \int_n^{n+1} u^{-(s+1)} \, du = x^{1-s} - \sum_{n \geq x} s \int_n^{n+1} [u]u^{-(s+1)} \, du = \int_x^\infty (s-1)u^{-s} - s[u]u^{-(s+1)} \, du = - \int_x^\infty u^{-s} \, du + s \int_x^\infty (u-[u])u^{-(s+1)} \, du = \frac{x^{1-s}}{1-s} + s \int_x^\infty (u-[u])u^{-(s+1)} \, du.
\]

Therefore

\[
\zeta(s) = \sum_{n \leq x} \frac{1}{n^s} + \frac{x^{1-s}}{s-1} - s \int_x^\infty (u-[u])u^{-(s+1)} \, du
\]

whenever \( \sigma > 1 \). However, we note that the three summands on the right are all analytic whenever \( \sigma > 0 \) except for the obvious pole at \( s = 1 \). This completes the proof.

**Lemma 2:** For all \( s \) with \( |s| \geq 2 \) and \( \sigma \geq 1/2 \)

\[
|\zeta(s)| \ll |s|^{1/2},
\]

where the implicit constant is absolute.
Lemma 2 and estimating crudely via Stirling's Formula

We get the third inequality since \(|t| \geq \sqrt{7}/2\), which follows from \(|s| \geq 2\) and \(\sigma \leq 3/2\).

Second,

\[
\frac{x^{1-s}}{s-1} \leq \frac{x^{1-\sigma}}{|s|-1} \leq x^{1-\sigma} \ll |s|^{1/2}.
\]

Finally,

\[
|s| \int_x^\infty (u-[u]) u^{-(s+1)} \, du \leq |s| \int_x^\infty u^{-\sigma-1} \, du \leq 2|s|x^{-1/2} \ll |s|^{1/2}.
\]

Proposition 1: The function \(\xi\) is entire and satisfies the functional equation \(\xi(s) = \xi(1-s)\). Further, for all \(R\) sufficiently large \(|\xi(s)| \leq \exp(R^{3/2})\) on the closed disc of radius \(R\).

Proof: Clearly the only possible poles of \(\xi\) come from the poles of \(\zeta(s)\) and \(\Gamma(s/2)\). But \(\zeta(s)\) has only the one simple pole at \(s = 1\) with residue 1, so the \(s - 1\) factor in \(\xi(s)\) takes care of that. The simple poles of \(\Gamma(s/2)\) at \(s = 0, -2, -4, \ldots\) are similarly cancelled by the zeros of \(s\zeta(s)\). Thus \(\xi(s)\) is entire.

The functional equation follows from that of the zeta function:

\[
\xi(s) = \frac{1}{2} s(s-1) \zeta(s) \Gamma(s/2) \pi^{-s/2} = \frac{1}{2} (1 - (1 - s))(s-1)\pi^{-(1-s)/2} \Gamma((1-s)/2) \zeta(1-s) = \xi(1-s).
\]

Finally, by the maximum modulus principle it suffices to consider those \(s\) with \(|s| = R\) for the final part of the proposition. Suppose that \(|s| \geq 30\). Then in particular \(|s| \geq 10\) and \(\log |s| > \pi\). We first consider the case where \(\sigma \geq 1/2\) (note that this implies that the argument of \(s/2\) is between \(-\pi/2\) and \(\pi/2\)). Then by Lemma 2 and estimating crudely via Stirling’s Formula

\[
|\xi(s)| \ll |s|^2 |\zeta(s)||\Gamma(s/2)| \pi^{-\sigma/2} \ll |s|^2 |s|^{1/2} |s^{(s-1)/2}| = \exp \left( (5/2) \log |s| \right) \exp \left( (1/2) \Re((s-1) \log s) \right) < \exp \left( (5/2) \log |s| \right) \exp \left( (1/2)(\sigma - 1 + |t|/2) \log |s| \right) < \exp \left( (1/4)|s| \log |s| \right) \exp \left( (3/4)|s| \log |s| \right) = \exp \left( |s| \log |s| \right).
\]
In the case where $\sigma < 1/2$ we consider $\xi(1-s)$ instead. Noting that $|1-s| \geq 10$, $\log|1-s| > \pi$, and $\Re(1-s) = 1 - \sigma \geq 1/2$, we see via the functional equation and estimating exactly as above with $1-s$ in place of $s$ that $|\xi(s)| = |\xi(1-s)| \ll \exp((1-s)|\log(1-s)|) \leq \exp(2|s|\log|s|)$, with the same implicit constant as above. Therefore for $R \geq 30$ we have $|\xi(s)| \leq C \exp(2R\log R) = \exp(2R\log R + \log C)$ for all $|s| = R$ and some constant $C > 1$. Now for $R \geq C$ sufficiently large, $R^{1/2} \geq 3\log R \geq 2\log R + \log C$, so that $|\xi(s)| \leq \exp(R^{3/2})$ whenever $|s| = R$.

**Theorem 1:** There is a constant $A$ such that

$$\xi(s) = \frac{1}{2} e^{As} \prod_{\rho} \left(1 - \frac{s}{\rho}\right) e^{s/\rho},$$

where the product is over all zeros $\rho$ of $\xi(s)$ in the critical strip. Moreover, this product converges uniformly on compact sets.

**Proof:** We first consider the zeros of $\xi$. Note that the $s(s-1)$ factor is used to cancel poles from the $\zeta(s)\Gamma(s/2)$ factor. Since $\Gamma(s/2)$ has no zeros, the zeros of $\xi$ must come from zeros of the zeta function. But we used the “trivial” zeros of the zeta function (at $s = -2, -4, \ldots$) to cancel poles of the Gamma factor. Thus the zeros of $\xi$ are precisely the zeros of the zeta function in the critical strip.

By exercises 18 and 27

$$\xi(0) = -\zeta(0) \lim_{s \to 0} (s/2)\Gamma(s/2) = 1/2.$$

Let $R$ be “sufficiently large” as per Proposition 1 and let $N(R)$ denote the number of zeros of $\xi$ in the disc of radius $R$ centered at the origin. By Jensen’s Inequality (with $R = 2R$ and $r = R$ there)

$$N(R) \leq \frac{\log \exp((2R)^{3/2}) - \log |\xi(0)|}{\log 2} \leq 3R^{3/2}. \quad (1)$$

From this we get

$$\sum_{R < |\rho| \leq 2R} \frac{1}{|\rho|^2} \leq \frac{1}{R^2} N(2R) \ll R^{-1/2}. \quad (2)$$

In particular,

$$\sum_{|\rho| > R} \frac{1}{|\rho|^2} = \sum_{n \geq 0} \sum_{2^n R < |\rho| \leq 2^{n+1} R} \frac{1}{|\rho|^2} \ll \sum_{n \geq 0} (2^n R)^{-1/2} \ll R^{-1/2}. \quad (2)$$

Since

$$(1-z)e^z = (1-z) \left(1 + z + \frac{z^2}{2!} + \cdots \right) = 1 + O(|z|^2)$$
uniformly for $|z| \leq 1$, we see that the infinite product
\[ f(s) := \prod_{\rho} \left( 1 - \frac{s}{\rho} \right) e^{s/\rho} \]
represents an entire function and is uniformly convergent on compact sets. Set
\[ g(s) := \frac{2\xi(s)}{f(s)}. \]
Then this is an entire function with no zeros and $g(0) = 1$.

Suppose for the moment that for all $R$ sufficiently large
\[ \max_{|s| \leq R} \{|g(s)|\} \leq \exp(c R^{3/2} \log R) \]
for some positive $c$ independent of $R$. Then we may apply the Borel-Carathéodory Lemma to the entire function $\log g(s)$ using $\Re\left\{ \log g(s) \right\} \ll R^{3/2} \log R$ on the closed disc of radius $R$ and $r = R/2$, giving
\[ \max_{|s| \leq R/2} |\log g(s)| \ll R^{3/2} \log R \cdot \frac{R}{R} < R^{11/6}. \]
Writing $\log g(s) = \sum_{n \geq 1} a_n s^n$ (since $g(0) = 1$), we clearly have
\[ \max_{|s| \leq R/2} \left| \sum_{n \geq 2} a_n s^n \right| \leq |a_1|R/2 + \max_{|s| \leq R/2} |\log g(s)| \ll R^{11/6}, \]
so that
\[ \lim_{|s| \to \infty} \frac{\sum_{n \geq 2} a_n s^n}{s^2} = 0. \]
But this implies that the entire function $\sum_{n \geq 2} a_n s^{n-2}$ is identically zero, thus $a_n = 0$ for all $n \geq 2$ and whence $\log g(s) = a_1 s$. Now setting $A = a_1$ yields the theorem. Hence, our proof is complete with the following.

Lemma 3: For $R$ sufficiently large inequality (3) above holds.

Proof: With the notation above, write
\[ f(s) = \prod_{|\rho| \leq R/2} \left( 1 - \frac{s}{\rho} \right) e^{s/\rho} \prod_{R/2 < |\rho| \leq 4R} \left( 1 - \frac{s}{\rho} \right) e^{s/\rho} \prod_{|\rho| > 4R} \left( 1 - \frac{s}{\rho} \right) e^{s/\rho} := h_1(s) h_2(s) h_3(s). \]
Suppose $R \leq |s| \leq 2R$ and $|\rho| \leq R/2$. Then $|1 - s/\rho| \geq |s|/|\rho| - 1 \geq 1$ and $|e^{s/\rho}| \geq e^{-2R/|\rho|}$. We see that
\[ |h_1(s)| \geq \prod_{|\rho| \leq R/2} e^{-2R/|\rho|} = \exp \left( -2R \sum_{|\rho| \leq R/2} \frac{1}{|\rho|} \right). \]
Since \( N(r) \ll r^{3/2} \) for all \( r \geq 1 \) by (1), we get for all \( r \geq 1 \)

\[
\sum_{|\rho| \leq r} \frac{1}{|\rho|} \ll \sum_{n=0}^{\left\lceil \log_2 r \right\rceil+1} \sum_{2^n \leq |\rho| < 2^{n+1}} \frac{1}{|\rho|} \\
\leq \sum_{n=0}^{\left\lceil \log_2 r \right\rceil+1} \frac{1}{2^n} \\
\leq \sum_{n=0}^{\left\lceil \log_2 r \right\rceil+1} \frac{2^{3n/2}}{2^n} \\
= \sum_{n=0}^{\left\lceil \log_2 r \right\rceil+1} 2^{n/2} \\
\ll 2^{(\log r)/2} \\
= r^{1/2}.
\]

In particular, we get

(4) \[ |h_1(s)| \geq \exp(-c_1 R^{3/2}) \]

for some positive \( c_1 \) independent of \( R \); this holds for all \( R \) sufficiently large and all \( s \) with \( R \leq |s| \leq 2R \).

We next consider the following collection of nonintersecting subintervals of \([R, 2R] \):

\[ [R, R + 1/R^2), [R + 1/R^2, R + 2/R^2), \ldots , [R + ([R^3] - 1)/R^2, R + [R^3]/R^2). \]

There are \([R^3] > 3R^{3/2} \) such subintervals for \( R \) sufficiently large and \( N(2R) \leq (2R)^{3/2} < 3R^{3/2} \) by (1). We conclude that for at least one of the intervals above there is no zero \( \rho \) with \( |\rho| \) in the interval. Let \( r \) be the midpoint of one such interval. Then \( |r - \rho| \geq 1/2R^2 \) for all of the zeros \( \rho \). Now for \( |s| = r \) and \( |\rho| \geq R/2 \) we have

\[
\left| 1 - \frac{s}{\rho} \right| = \frac{|r - \rho|}{|\rho|} \geq R^{-3}.
\]

Thus for \( |s| = r \) and \( R \) sufficiently large (recall \( r \in [R, 2R] \))

(5) \[ |h_2(s)| \geq \prod_{R/2 < |\rho| \leq 4R} R^{-3} e^{R^3(s/\rho)} \\
\geq \prod_{R/2 < |\rho| \leq 4R} R^{-3} e^{-r/|\rho|} \\
\geq \prod_{R/2 < |\rho| \leq 4R} R^{-3} e^{-2r/R} \\
\geq \prod_{R/2 < |\rho| \leq 4R} R^{-3} e^{-4} \\
\geq R^{-4N(4R)} \\
\geq \exp\left(-4(4R)^{3/2} \log R\right) \\
= \exp\left(-c_2 R^{3/2} \log R\right)
\]
by (1) once more, with \( c_2 = 32 = 4^{5/2} \).

Finally, suppose \(|s| \in [R, 2R]\) and \(|\rho| > 4R\). One readily verifies that

\[
(1 - z)e^z = 1 - \sum_{n \geq 2} \frac{(n - 1)z^n}{n!}.
\]

Thus

\[
|(1 - z)e^z| = \left| 1 - \sum_{n \geq 2} \frac{(n - 1)z^n}{n!} \right|
\]

\[
\geq 1 - \sum_{n \geq 2} \frac{(n - 1)|z|^n}{n!}
\]

\[
> 1 - \frac{|z|^2}{2} - \frac{1}{3} \sum_{n \geq 3} |z|^n
\]

\[
= 1 - \frac{|z|^2}{2} - \frac{1}{3} |z|^3
\]

\[
> 1 - \frac{|z|^2}{2} - \frac{2}{3} |z|^3
\]

\[
> 1 - \frac{|z|^2}{2} - \frac{1}{3} |z|^2
\]

\[
= 1 - \frac{5|z|^2}{6}
\]

\[
> 1 - |z|^2 + |z|^4/2
\]

\[
> \sum_{n \geq 0} \frac{(-1)^n|z|^{2n}}{n!}
\]

\[
= e^{-|z|^2}.
\]

Now by (2), for all \(|s| \in [R, 2R]\) we have with \( z = s/\rho \)

\[
|h_3(s)| = \prod_{|\rho| > 4} |(1 - z)e^z|
\]

\[
\geq \prod_{|\rho| > 4} e^{-|s/\rho|^2}
\]

\[
\geq \prod_{|\rho| > 4R} e^{-4R^2/|\rho|^2}
\]

\[
(6)
\]

\[
= \exp \left( -4R^2 \sum_{|\rho| > 4R} \frac{1}{|\rho|^2} \right)
\]

\[
\geq \exp(-c_3R^{3/2})
\]

for some positive \( c_3 \) independent of \( R \).

Combining (4)-(6), there is an \( r \in [R, 2R] \) such that \(|f(s)| \geq \exp(-c_1R^{3/2} - c_2R^{3/2} \log R - c_3R^{3/2})\)

for all \(|s| = r\). Using this estimate together with the maximum modulus principle, Proposition 1, and the
definition of \( g(s) \), we see that

\[
\max_{|s| \leq R} \{|g(s)|\} \leq \max_{|s| = r} \{|g(s)|\}
\]

\[
= \max_{|s| = r} \{|g(s)|\}
\]

\[
= \max_{|s| = r} \left\{ \frac{|2\xi(s)|}{|f(s)|} \right\}
\]

\[
\leq \exp(2^{3/2}) \exp\left( - (c_1 + c_2 + c_3)R^{5/2} \log R \right)
\]

\[
\leq \exp(cR^{3/2} \log R),
\]

with \( c = 2^{5/2} + c_1 + c_2 + c_3 \).

**Corollary 1:** For \( A \) as in Theorem 1,

\[
\frac{\xi'(s)}{\xi(s)} = A + \sum_{\rho} \left( \frac{1}{s - \rho} + \frac{1}{\rho} \right).
\]

Proof: By Theorem 1

\[
\log \xi(s) = -\log 2 + As + \sum_{\rho} \log(1 - s/\rho) + s/\rho
\]

uniformly on compact sets. We therefore may differentiate term-by-term to get Corollary 1.

**Corollary 2:** For \( A \) as in Theorem 1,

\[
\frac{\zeta'(s)}{\zeta(s)} = A + \sum_{\rho} \left( \frac{1}{s - \rho} + \frac{1}{\rho} \right) + \frac{1}{2} \log \pi - \frac{1}{s - 1} - \frac{\Gamma'(1+s/2)}{2\Gamma(1+s/2)}.
\]

Proof: By the definition of \( \xi(s) \) and properties of the Gamma function \( \xi(s) = (s-1)\xi(s)\Gamma(1+s/2)\pi^{-s/2} \), so that

\[
\log \xi(s) = \log \zeta(s) - \log \pi \frac{s}{2} + \log(s - 1) + \log \Gamma(1 + s/2).
\]

Differentiating and applying Corollary 1 gives the result.

We make the following definition, which changes somewhat the notation established in the proof of Theorem 1 above.

**Definition:** For a positive \( T \) let \( N(T) \) denote the number of zeros \( \rho \) of \( \zeta(s) \) in the critical strip with \( 0 < \Im(\rho) \leq T \), where any zeros with \( \Im(\rho) = T \) are counted with half weight.

**Theorem 2:** For all positive \( T \)

\[
N(T + 1) - N(T) \ll \log(T + 2),
\]

where the implicit constant is absolute.
Proof: Apply Jensen’s Inequality to \( F(s) = \xi(s + 2 + i(T + 1/2)) \) with \( R = 11/6 \) and \( r = 7/4 \). Note that the rectangle \( \{a + ib : 1/2 \leq a \leq 1, \ T \leq b \leq T + 1 \} \) is entirely within the disc of radius \( r \) centered at \( 2 + i(T + 1/2) \). Assuming \( T \geq 4 \), say, for those \( s \) in the disc of radius \( R \) with \( \sigma \geq -3/2 \) (so that \( 2 + \sigma \geq 1/2 \)) we have \( |\xi(s + 2 + i(T + 1/2))| \ll T^2 T^{1/2} T^{(R+1)/2} = T^{47/12} \) exactly as in the estimates in the proof of Proposition 1. The same upper bound holds if \( 2 + \sigma \leq 1/2 \) by the functional equation for \( \xi(s) \). Therefore by Jensen’s Inequality we infer that the number of zeros in the rectangle above is \( O(\log T) \) assuming that \( T \geq 4 \). If \( T \leq 4 \) then the number of zeros in the rectangle is clearly \( O(1) = O(\log(T + 2)) \). But \( \rho \) is a zero of \( \xi(s) \) if and only if \( 1 - \rho \) is also a zero, so that the rectangle \( \{a + ib : 0 \leq a \leq 1/2, T \leq b \leq T + 1 \} \) contains the exact same number of zeros as the rectangle above. Theorem 2 follows.

**Theorem 3:** For all \( s = \sigma + it \) with \(-1 \leq \sigma \leq 2 \) we have

\[
\frac{\zeta'(s)}{\zeta(s)} = \frac{-1}{s-1} + \sum_{\rho} \frac{1}{s-\rho} + O(\log(|t| + 4))
\]
uniformly.

Proof: Applying Lemma 2 from the handout on Stirling’s Formula together with Corollary 2 to Theorem 1 gives

\[
\frac{\zeta'(s)}{\zeta(s)} = \frac{-1}{s-1} + \sum_{\rho} \frac{1}{s-\rho} + O(\log(|t| + 4)).
\]

As noted earlier

\[
\left| \frac{\zeta'(2 + it)}{\zeta(2 + it)} \right| \leq \sum_{n \geq 1} \frac{\Lambda(n)}{n^2} < \zeta(2).
\]

Now substituting \( 2 + it \) into our first equation yields

\[
\frac{-1}{1 + it} + \sum_{\rho} \frac{1}{2 + it - \rho} + \frac{1}{\rho} = O(\log(|t| + 4)).
\]

Thus

\[
(7) \quad \frac{\zeta'(s)}{\zeta(s)} = \frac{-1}{s-1} + \sum_{\rho} \frac{1}{s-\rho} - \frac{1}{2 + it - \rho} + O(\log(|t| + 4)).
\]

Now by Theorem 2

\[
\left| \sum_{\rho} \frac{1}{2 + it - \rho} \right| = O \left( \sum_{\rho} \frac{1}{\rho} \right) = O(\log(|t| + 4)).
\]

Therefore

\[
\sum_{\rho} \frac{1}{s - \rho} - \frac{1}{2 + it - \rho} = \sum_{\rho} \frac{1}{s - \rho} - \frac{1}{2 + it - \rho} + \sum_{n \geq 1} \sum_{\rho} \frac{1}{s - \rho} - \frac{1}{2 + it - \rho} \]

\[
= \sum_{\rho} \frac{1}{s - \rho} + O(\log(|t| + 4)) + \sum_{n \geq 1} \sum_{\rho} \frac{1}{s - \rho} - \frac{1}{2 + it - \rho}.
\]

(8)
Note that if \( n < |\Re(\rho) - t| \leq n + 1 \), then
\[
\frac{1}{s - \rho} - \frac{1}{2 + it - \rho} = \frac{2 - \sigma}{(s - \rho)(2 + it - \rho)} = O \left( \frac{1}{n^2} \right).
\]

Using this, we see via Theorem 2 once more that
\[
\sum_{n \geq 1} \sum_{n < |\Im(\rho) - t| \leq n + 1} \left( \frac{1}{s - \rho} - \frac{1}{2 + it - \rho} \right) = O \left( \sum_{n \geq 1} \frac{N(t + n + 1) - N(t + n) + N(t - n) - N(t - n - 1)}{n^2} \right)
= O \left( \sum_{n \geq 1} \frac{\log(|t| + 4 + n)}{n^2} \right)
= O \left( \log(|t| + 4) \right).
\]

Theorem 3 follows from (7)-(9).

**Theorem 4:** For all \( T \geq 2 \) there is a \( T_1 \in [T, T + 1] \) such that
\[
\left| \frac{\zeta'(\sigma + iT_1)}{\zeta(\sigma + iT_1)} \right| \ll (\log T)^2
\]
uniformly for all \(-1 \leq \sigma \leq 2\).

Proof: By Theorem 2 \( N(T + 1) - N(T) \ll \log T \). Thus by the pigeon-hole principle there is a \( T_1 \in [T, T + 1] \) such that \( |T_1 - \Re(\zeta(\rho))| \gg 1/\log T \) for all zeros \( \rho \) in the critical strip. Theorem 4 follows from this and Theorem 3.

**Proposition 2:** Suppose \( \sigma \leq -1 \) and either \( \sigma \) is an odd integer or \(|t| \geq 2\). Then
\[
\left| \frac{\zeta'(s)}{\zeta(s)} \right| \ll \log(|s| + 4).
\]

Proof: By exercise 18 \( \zeta(s) = \zeta(1-s)2^{s}\pi^{s-1}\Gamma(1-s)\sin(\pi s/2) \). Therefore
\[
(10) \quad \frac{\zeta'(s)}{\zeta(s)} = -\frac{\zeta'(1-s)}{\zeta(1-s)} + \log(2\pi) - \frac{\Gamma'(1-s)}{\Gamma(1-s)} + \frac{2}{\pi} \cot(\pi s/2).
\]

Now \( \Re(1-s) = 1 - \sigma \geq 2 \), so that as before
\[
(11) \quad \left| \frac{\zeta'(1-s)}{\zeta(1-s)} \right| \ll 1.
\]

By Lemma 2 of the handout on Stirling’s Formula (note that the argument of \( 1-s \) is between \(-\pi/2\) and \(\pi/2\))
\[
(12) \quad \left| \frac{\Gamma'(1-s)}{\Gamma(1-s)} \right| = \log(1-s) + O(1/|s-1|) \ll \log(|s| + 4).
\]

Finally, since \( \pi s/2 \) is bounded away from the poles of the cotangent function by hypothesis
\[
(13) \quad |\cot(\pi s/2)| = \left| i + \frac{2i}{e^{i\pi s} - 1} \right| \ll 1.
\]

Proposition 2 follows from (10)-(13).