Math 680 Fall 2017

Properties of the Xi Function and Consequences Thereof

We saw earlier the xi function $\xi(s) = \frac{1}{2}s(s-1)\zeta(s)\Gamma(s/2)\pi^{-s/2}$. Our goal is to find an infinite product representation of this function. With that, we'll get some nice corollaries.

Lemma 1: Suppose x is a positive integer and $s = \sigma + it \neq 1$ with $\sigma > 0$. Then

$$\zeta(s) = \sum_{n \le x} \frac{1}{n^s} + \frac{x^{1-s}}{s-1} - s\int_x^\infty \big(u - [u]\big)u^{-(s+1)}\,du.$$

Proof: First assume that $\sigma > 1$ and write $\zeta(s) = \sum_{n\le x} \frac{1}{n^s} + \sum_{n>x} \frac{1}{n^s}$. Now

$$\begin{aligned}
-\sum_{n>x} \frac{1}{n^s} &= \sum_{n\ge x} -(n+1)^{-s} \\
&= \sum_{n\ge x} (n+1)(n+1)^{-s} - (n+1)^{-s} - (n+1)^{1-s} \\
&= \sum_{n\ge x} n(n+1)^{-s} - (n+1)^{1-s} \\
&= x^{1-s} + \sum_{n\ge x} n(n+1)^{-s} - n^{1-s} \\
&= x^{1-s} + \sum_{n\ge x} n\big((n+1)^{-s} - n^{-s}\big) \\
&= x^{1-s} - \sum_{n\ge x} ns\int_n^{n+1} u^{-(s+1)}\,du \\
&= x^{1-s} - \sum_{n\ge x} s\int_n^{n+1} [u]u^{-(s+1)}\,du \\
&= \int_x^\infty (s-1)u^{-s} - s[u]u^{-(s+1)}\,du \\
&= -\int_x^\infty u^{-s}\,du + s\int_x^\infty \big(u-[u]\big)u^{-(s+1)}\,du \\
&= \frac{x^{1-s}}{1-s} + s\int_x^\infty \big(u-[u]\big)u^{-(s+1)}\,du.
\end{aligned}$$

Therefore

$$\zeta(s) = \sum_{n \le x} \frac{1}{n^s} + \frac{x^{1-s}}{s-1} - s\int_x^\infty \big(u - [u]\big)u^{-(s+1)}\,du$$

whenever $\sigma > 1$. However, we note that the three summands on the right are all analytic whenever $\sigma > 0$ except for the obvious pole at $s = 1$. This completes the proof.

Lemma 2: For all s with $|s| \ge 2$ and $\sigma \ge 1/2$

$$|\zeta(s)| \ll |s|^{1/2},$$

where the implicit constant is absolute.

Proof: Via the usual infinite sum representation we see that $|\zeta(s)| \ll 1$ whenever $\sigma \geq 3/2$, say, so we need only concern ourselves with the case where $1/2 \leq \sigma < 3/2$.

Suppose s satisfies $|s| \geq 2$ and $1/2 \leq \sigma < 3/2$. We apply Lemma 1 with $x = [1 + |t|]$. First,

$$\left| \sum_{n \leq x} \frac{1}{n^s} \right| \leq \sum_{n \leq x} \frac{1}{n^\sigma} \ll x^{1-\sigma} \ll |t|^{1/2} < |s|^{1/2}.$$

(We get the third inequality since $|t| \geq \sqrt{7}/2$, which follows from $|s| \geq 2$ and $\sigma \leq 3/2$.) Second,

$$\left| \frac{x^{1-s}}{s-1} \right| \leq \frac{x^{1-\sigma}}{|s| - 1} \leq x^{1-\sigma} \ll |s|^{1/2}.$$

Finally,

$$\begin{aligned} \left| s \int_x^\infty \big(u - [u]\big) u^{-(s+1)}\, du \right| &\leq |s| \int_x^\infty u^{-\sigma - 1}\, du \\ &\leq |s| \int_x^\infty u^{-3/2}\, du \\ &= 2|s| x^{-1/2} \\ &\ll |s|^{1/2}. \end{aligned}$$

Proposition 1: The function ξ is entire and satisfies the functional equation $\xi(s) = \xi(1-s)$. Further, for all R sufficiently large $|\xi(s)| \leq \exp(R^{3/2})$ on the closed disc of radius R.

Proof: Clearly the only possible poles of ξ come from the poles of $\zeta(s)$ and $\Gamma(s/2)$. But $\zeta(s)$ has only the one simple pole at $s = 1$ with residue 1, so the $s - 1$ factor in $\xi(s)$ takes care of that. The simple poles of $\Gamma(s/2)$ at $s = 0, -2, -4, \ldots$ are similarly cancelled by the zeros of $s\zeta(s)$. Thus $\xi(s)$ is entire.

The functional equation follows from that of the zeta function:

$$\begin{aligned} \xi(s) &= \frac{1}{2} s(s-1)\zeta(s)\Gamma(s/2)\pi^{-s/2} \\ &= \frac{1}{2}\big(1 - (1-s)\big)(s-1)\pi^{-(1-s)/2}\Gamma\big((1-s)/2\big)\zeta(1-s) \\ &= \xi(1-s). \end{aligned}$$

Finally, by the maximum modulus principle it suffices to consider those s with $|s| = R$ for the final part of the proposition. Suppose that $|s| \geq 30$. Then in particular $|s| \geq 10$ and $\log|s| > \pi$. We first consider the case where $\sigma \geq 1/2$ (note that this implies that the argument of $s/2$ is between $-\pi/2$ and $\pi/2$). Then by Lemma 2 and estimating crudely via Stirling's Formula

$$\begin{aligned} |\xi(s)| &\ll |s|^2 |\zeta(s)| |\Gamma(s/2)| \pi^{-\sigma/2} \\ &\ll |s|^2 |s|^{1/2} |s^{(s-1)/2}| \\ &= \exp\big((5/2)\log|s|\big) \exp\big((1/2)\Re((s-1)\log s)\big) \\ &< \exp\big((5/2)\log|s|\big) \exp\big((1/2)(\sigma - 1 + |t|/2)\log|s|\big) \\ &< \exp\big((1/4)|s|\log|s|\big) \exp\big((3/4)|s|\log|s|\big) \\ &= \exp\big(|s|\log|s|\big). \end{aligned}$$

In the case where $\sigma < 1/2$ we consider $\xi(1-s)$ instead. Noting that $|1-s| \geq 10$, $\log|1-s| > \pi$, and $\Re(1-s) = 1-\sigma \geq 1/2$, we see via the functional equation and estimating exactly as above with $1-s$ in place of s that $|\xi(s)| = |\xi(1-s)| \ll \exp\left(|1-s|\log|1-s|\right) \leq \exp\left(2|s|\log|s|\right)$, with the same implicit constant as above. Therefore for $R \geq 30$ we have $|\xi(s)| \leq C\exp(2R\log R) = \exp(2R\log R + \log C)$ for all $|s| = R$ and some constant $C > 1$. Now for $R \geq C$ sufficiently large, $R^{1/2} \geq 3\log R \geq 2\log R + \log C$, so that $|\xi(s)| \leq \exp(R^{3/2})$ whenever $|s| = R$.

Theorem 1: There is a constant A such that

$$\xi(s) = \frac{1}{2}e^{As}\prod_{\rho}\left(1-\frac{s}{\rho}\right)e^{s/\rho},$$

where the product is over all zeros ρ of $\zeta(s)$ in the critical strip. Moreover, this product converges uniformly on compact sets.

Proof: We first consider the zeros of ξ. Note that the $s(s-1)$ factor is used to cancel poles from the $\zeta(s)\Gamma(s/2)$ factor. Since $\Gamma(s/2)$ has no zeros, the zeros of ξ must come from zeros of the zeta function. But we used the "trivial" zeros of the zeta function (at $s = -2, -4, \dots$) to cancel poles of the Gamma factor. Thus the zeros of ξ are precisely the zeros of the zeta function in the critical strip.

By exercises 18 and 27

$$\xi(0) = -\zeta(0)\lim_{s\to 0}(s/2)\Gamma(s/2) = 1/2.$$

Let R be "sufficently large" as per Proposition 1 and let $N(R)$ denote the number of zeros of ξ in the disc of radius R centered at the origin. By Jensen's Inequality (with $R = 2R$ and $r = R$ there)

$$N(R) \leq \frac{\log\exp((2R)^{3/2}) - \log|\xi(0)|}{\log 2} \leq 3R^{3/2}. \tag{1}$$

From this we get

$$\sum_{R<|\rho|\leq 2R}\frac{1}{|\rho|^2} \leq \frac{1}{R^2}N(2R) \ll R^{-1/2}.$$

In particular,

$$\begin{aligned}\sum_{|\rho|>R}\frac{1}{|\rho|^2} &= \sum_{n\geq 0}\sum_{2^nR<|\rho|\leq 2^{n+1}R}\frac{1}{|\rho|^2}\\ &\ll \sum_{n\geq 0}(2^nR)^{-1/2}\\ &\ll R^{-1/2}.\end{aligned} \tag{2}$$

Since

$$(1-z)e^z = (1-z)\left(1+z+\frac{z^2}{2!}+\cdots\right) = 1 + O(|z|^2)$$

uniformly for $|z| \leq 1$, we see that the infinite product

$$f(s) := \prod_{\rho} \left(1 - \frac{s}{\rho}\right) e^{s/\rho}$$

represents an entire function and is uniformly convergent on compact sets. Set

$$g(s) := \frac{2\xi(s)}{f(s)}.$$

Then this is an entire function with *no* zeros and $g(0) = 1$.

Suppose for the moment that for all R sufficiently large

$$\max_{|s|\leq R} \{|g(s)|\} \leq \exp(cR^{3/2} \log R) \tag{3}$$

for some positive c independent of R. Then we may apply the Borel-Carathéodory Lemma to the entire function $\log g(s)$ using $\Re\big(\log g(s)\big) \ll R^{3/2} \log R$ on the closed disc of radius R and $r = R/2$, giving

$$\max_{|s|\leq R/2} |\log g(s)| \ll R^{3/2} \log R \cdot \frac{R}{R} < R^{11/6}.$$

Writing $\log g(s) = \sum_{n\geq 1} a_n s^n$ (since $g(0) = 1$), we clearly have

$$\max_{|s|\leq R/2} \left| \sum_{n\geq 2} a_n s^n \right| \leq |a_1|R/2 + \max_{|s|\leq R/2} |\log g(s)| \ll R^{11/6},$$

so that

$$\lim_{|s|\to\infty} \frac{\sum_{n\geq 2} a_n s^n}{s^2} = 0.$$

But this implies that the entire function $\sum_{n\geq 2} a_n s^{n-2}$ is identically zero, thus $a_n = 0$ for all $n \geq 2$ and whence $\log g(s) = a_1 s$. Now setting $A = a_1$ yields the theorem. Hence, our proof is complete with the following.

Lemma 3: For R sufficiently large inequality (3) above holds.

Proof: With the notation above, write

$$\begin{aligned} f(s) &= \prod_{|\rho|\leq R/2} \left(1 - \frac{s}{\rho}\right) e^{s/\rho} \prod_{R/2<|\rho|\leq 4R} \left(1 - \frac{s}{\rho}\right) e^{s/\rho} \prod_{|\rho|>4R} \left(1 - \frac{s}{\rho}\right) e^{s/\rho} \\ &:= h_1(s)h_2(s)h_3(s). \end{aligned}$$

Suppose $R \leq |s| \leq 2R$ and $|\rho| \leq R/2$. Then $|1 - s/\rho| \geq |s|/|\rho| - 1 \geq 1$ and $|e^{s/\rho}| \geq e^{-2R/|\rho|}$. We see that

$$|h_1(s)| \geq \prod_{|\rho|\leq R/2} e^{-2R/|\rho|} = \exp\left(-2R \sum_{|\rho|\leq R/2} \frac{1}{|\rho|}\right).$$

Since $N(r) \ll r^{3/2}$ for all $r \geq 1$ by (1), we get for all $r \geq 1$

$$\begin{aligned}\sum_{|\rho|\leq r} \frac{1}{|\rho|} &\ll \sum_{n=0}^{[\log_2 r]+1} \sum_{2^n \leq |\rho| < 2^{n+1}} \frac{1}{|\rho|} \\ &\leq \sum_{n=0}^{[\log_2 r]+1} \sum_{|\rho|\leq 2^{n+1}} \frac{1}{2^n} \\ &\ll \sum_{n=0}^{[\log_2 r]+1} \frac{2^{3n/2}}{2^n} \\ &= \sum_{n=0}^{[\log_2 r]+1} 2^{n/2} \\ &\ll 2^{(\log r)/2} \\ &= r^{1/2}.\end{aligned}$$

In particular, we get

$$|h_1(s)| \geq \exp(-c_1 R^{3/2}) \tag{4}$$

for some positive c_1 independent of R; this holds for all R sufficiently large and all s with $R \leq |s| \leq 2R$.

We next consider the following collection of nonintersecting subintervals of $[R, 2R]$:

$$[R, R+1/R^2),\ [R+1/R^2, R+2/R^2), \dots,\ [R+([R^3]-1)/R^2, R+[R^3]/R^2).$$

There are $[R^3] > 3R^{3/2}$ such subintervals for R sufficiently large and $N(2R) \leq (2R)^{3/2} < 3R^{3/2}$ by (1). We conclude that for at least one of the intervals above there is no zero ρ with $|\rho|$ in the interval. Let r be the midpoint of one such interval. Then $|r-\rho| \geq 1/2R^2$ for all of the zeros ρ. Now for $|s| = r$ and $|\rho| \geq R/2$ we have

$$\left|1 - \frac{s}{\rho}\right| = \frac{|r-\rho|}{|\rho|} \geq R^{-3}.$$

Thus for $|s| = r$ and R sufficiently large (recall $r \in [R, 2R]$)

$$\begin{aligned}|h_2(s)| &\geq \prod_{R/2<|\rho|\leq 4R} R^{-3} e^{\Re(s/\rho)} \\ &\geq \prod_{R/2<|\rho|\leq 4R} R^{-3} e^{-r/|\rho|} \\ &\geq \prod_{R/2<|\rho|\leq 4R} R^{-3} e^{-2r/R} \\ &\geq \prod_{R/2<|\rho|\leq 4R} R^{-3} e^{-4} \\ &\geq R^{-4N(4R)} \\ &\geq \exp\big(-4(4R)^{3/2}\log R\big) \\ &= \exp\big(-c_2 R^{3/2}\log R\big)\end{aligned} \tag{5}$$

by (1) once more, with $c_2 = 32 = 4^{5/2}$.

Finally, suppose $|s| \in [R, 2R]$ and $|\rho| > 4R$. One readily verifies that

$$(1-z)e^z = 1 - \sum_{n\geq 2} \frac{(n-1)z^n}{n!}.$$

Thus

$$\begin{aligned}
|(1-z)e^z| &= \left| 1 - \sum_{n\geq 2} \frac{(n-1)z^n}{n!} \right| \\
&\geq 1 - \sum_{n\geq 2} \frac{(n-1)|z|^n}{n!} \\
&> 1 - \frac{|z|^2}{2} - \frac{1}{3}\sum_{n\geq 3} |z|^n \\
&= 1 - \frac{|z|^2}{2} - \frac{1}{3}\frac{|z|^3}{1-|z|} \\
&> 1 - \frac{|z|^2}{2} - \frac{2}{3}|z|^3 \\
&> 1 - \frac{|z|^2}{2} - \frac{1}{3}|z|^2 \\
&= 1 - \frac{5|z|^2}{6} \\
&> 1 - |z|^2 + |z|^4/2 \\
&> \sum_{n\geq 0} \frac{(-1)^n |z|^{2n}}{n!} \\
&= e^{-|z|^2}.
\end{aligned}$$

Now by (2), for all $|s| \in [R, 2R]$ we have with $z = s/\rho$

$$\begin{aligned}
|h_3(s)| &= \prod_{|\rho|>4} |(1-z)e^z| \\
&\geq \prod_{|\rho|>4} e^{-|s/\rho|^2} \\
&\geq \prod_{|\rho|>4R} e^{-4R^2/|\rho|^2} \\
&= \exp\left(-4R^2 \sum_{|\rho|>4R} \frac{1}{|\rho|^2} \right) \\
&\geq \exp(-c_3 R^{3/2})
\end{aligned} \tag{6}$$

for some positive c_3 independent of R.

Combining (4)-(6), there is an $r \in [R, 2R]$ such that $|f(s)| \geq \exp(-c_1 R^{3/2} - c_2 R^{3/2}\log R - c_3 R^{3/2})$ for all $|s| = r$. Using this estimate together with the maximum modulus principle, Proposition 1, and the definition of $g(s)$, we see that

$$\begin{aligned}\max_{|s|\le R}\{|g(s)|\} &\le \max_{|s|\le r}\{|g(s)|\}\\ &= \max_{|s|=r}\{|g(s)|\}\\ &= \max_{|s|=r}\left\{\frac{|2\xi(s)|}{|f(s)|}\right\}\\ &< \frac{\exp(2r^{3/2})}{\exp\big(-(c_1+c_2+c3)R^{3/2}\log R\big)}\\ &\le \exp(cR^{3/2}\log R),\end{aligned}$$

with $c = 2^{5/2} + c_1 + c_2 + c_3$.

Corollary 1: For A as in Theorem 1,

$$\frac{\xi'(s)}{\xi(s)} = A + \sum_\rho \left(\frac{1}{s-\rho} + \frac{1}{\rho}\right).$$

Proof: By Theorem 1

$$\log \xi(s) = -\log 2 + As + \sum_\rho \log(1 - s/\rho) + s/\rho$$

uniformly on compact sets. We therefore may differentiate term-by-term to get Corollary 1.

Corollary 2: For A as in Theorem 1,

$$\frac{\zeta'(s)}{\zeta(s)} = A + \sum_\rho \left(\frac{1}{s-\rho} + \frac{1}{\rho}\right) + \frac{1}{2}\log\pi - \frac{1}{s-1} - \frac{\Gamma'(1+s/2)}{2\Gamma(1+s/2)}.$$

Proof: By the definition of $\xi(s)$ and properties of the Gamma function $\xi(s) = (s-1)\zeta(s)\Gamma(1+s/2)\pi^{-s/2}$, so that

$$\log \xi(s) = \log \zeta(s) - \log \pi\frac{s}{2} + \log(s-1) + \log\Gamma(1+s/2).$$

Differentiating and applying Corollary 1 gives the result.

We make the following definition, which changes somewhat the notation established in the proof of Theorem 1 above.

Definition: For a positive T let $N(T)$ denote the number of zeros ρ of $\zeta(s)$ in the critical strip with $0 < \Im(\rho) \le T$, where any zeros with $\Im(\rho) = T$ are counted with half weight.

Theorem 2: For all positive T

$$N(T+1) - N(T) \ll \log(T+2),$$

where the implicit constant is absolute.

Proof: Apply Jensen's Inequality to $F(s) = \xi\big(s + 2 + i(T + 1/2)\big)$ with $R = 11/6$ and $r = 7/4$. Note that the rectangle $\{a + ib : 1/2 \le a \le 1,\ T \le b \le T + 1\}$ is entirely within the disc of radius r centered at $2 + i(T + 1/2)$. Assuming $T \ge 4$, say, for those s in the disc of radius R with $\sigma \ge -3/2$ (so that $2 + \sigma \ge 1/2$) we have $|\xi\big(s + 2 + i(T + 1/2)\big)| \ll T^2 T^{1/2} T^{(R+1)/2} = T^{47/12}$ exactly as in the estimates in the proof of Proposition 1. The same upper bound holds if $2 + \sigma \le 1/2$ by the functional equation for $\xi(s)$. Therefore by Jensen's Inequality we infer that the number of zeros in the rectangle above is $O(\log T)$ assuming that $T \ge 4$. If $T \le 4$ then the number of zeros in the rectangle is clearly $O(1) = O\big(\log(T + 2)\big)$. But ρ is a zero of $\xi(s)$ if and only if $1 - \overline{\rho}$ is also a zero, so that the rectangle $\{a + ib : 0 \le a \le 1/2, T \le b \le T + 1\}$ contains the exact same number of zeros as the rectangle above. Theorem 2 follows.

Theorem 3: For all $s = \sigma + it$ with $-1 \le \sigma \le 2$ we have

$$\frac{\zeta'(s)}{\zeta(s)} = \frac{-1}{s-1} + \sum_{\substack{\rho \\ |\Im(\rho) - t| \le 1}} \frac{1}{s - \rho} + O\big(\log(|t| + 4)\big)$$

uniformly.

Proof: Applying Lemma 2 from the handout on Stirling's Formula together with Corollary 2 to Theorem 1 gives

$$\frac{\zeta'(s)}{\zeta(s)} = \frac{-1}{s-1} + \sum_{\rho} \frac{1}{s - \rho} + \frac{1}{\rho} + O\big(\log(|t| + 4)\big).$$

As noted earlier

$$\left|\frac{\zeta'(2 + it)}{\zeta(2 + it)}\right| \le \sum_{n \ge 1} \frac{\Lambda(n)}{n^2} < \zeta(2).$$

Now substituting $2 + it$ into our first equation yields

$$\frac{-1}{1 + it} + \sum_{\rho} \frac{1}{2 + it - \rho} + \frac{1}{\rho} = O\big(\log(|t| + 4)\big).$$

Thus

$$\frac{\zeta'(s)}{\zeta(s)} = \frac{-1}{s-1} + \sum_{\rho} \frac{1}{s - \rho} - \frac{1}{2 + it - \rho} + O\big(\log(|t| + 4)\big). \tag{7}$$

Now by Theorem 2

$$\left|\sum_{\substack{\rho \\ |\Im(\rho) - t| \le 1}} \frac{1}{2 + it - \rho}\right| = O\left(\sum_{\substack{\rho \\ |\Im(\rho) - t| \le 1}} 1\right) = O\big(\log(|t| + 4)\big).$$

Therefore

$$\begin{aligned} \sum_{\rho} \frac{1}{s - \rho} - \frac{1}{2 + it - \rho} &= \sum_{\substack{\rho \\ |\Im(\rho) - t| \le 1}} \frac{1}{s - \rho} - \frac{1}{2 + it - \rho} + \sum_{n \ge 1} \sum_{\substack{\rho \\ n < |\Im(\rho) - t| \le n+1}} \frac{1}{s - \rho} - \frac{1}{2 + it - \rho} \\ &= \sum_{\substack{\rho \\ |\Im(\rho) - t| \le 1}} \frac{1}{s - \rho} + O\big(\log(|t| + 4)\big) + \sum_{n \ge 1} \sum_{\substack{\rho \\ n < |\Im(\rho) - t| \le n+1}} \frac{1}{s - \rho} - \frac{1}{2 + it - \rho}. \end{aligned} \tag{8}$$

Note that if $n < |\Im(\rho) - t| \le n+1$, then

$$\frac{1}{s-\rho} - \frac{1}{2+it-\rho} = \frac{2-\sigma}{(s-\rho)(2+it-\rho)} = O\left(\frac{1}{n^2}\right).$$

Using this, we see via Theorem 2 once more that

(9)
$$\begin{aligned}\sum_{n\ge1} \sum_{\substack{\rho \\ n<|\Im(\rho)-t|\le n+1}} \frac{1}{s-\rho} - \frac{1}{2+it-\rho} &= O\left(\sum_{n\ge1} \frac{N(t+n+1)-N(t+n)+N(t-n)-N(t-n-1)}{n^2}\right) \\ &= O\left(\sum_{n\ge1} \frac{\log(|t|+4+n)}{n^2}\right) \\ &= O\big(\log(|t|+4)\big).\end{aligned}$$

Theorem 3 follows from (7)-(9).

Theorem 4: For all $T \ge 2$ there is a $T_1 \in [T, T+1]$ such that

$$\left|\frac{\zeta'(\sigma+iT_1)}{\zeta(\sigma+iT_1)}\right| \ll (\log T)^2$$

uniformly for all $-1 \le \sigma \le 2$.

Proof: By Theorem 2 $N(T+1) - N(T) \ll \log T$. Thus by the pigeon-hole principle there is a $T_1 \in [T, T+1]$ such that $|T_1 - \Im(\rho)| \gg 1/\log T$ for all zeros ρ in the critical strip. Theorem 4 follows from this and Theorem 3.

Proposition 2: Suppose $\sigma \le -1$ and either σ is an odd integer or $|t| \ge 2$. Then

$$\left|\frac{\zeta'(s)}{\zeta(s)}\right| \ll \log(|s|+4).$$

Proof: By exercise 18 $\zeta(s) = \zeta(1-s)2^s\pi^{s-1}\Gamma(1-s)\sin(\pi s/2)$. Therefore

$$\frac{\zeta'(s)}{\zeta(s)} = -\frac{\zeta'(1-s)}{\zeta(1-s)} + \log(2\pi) - \frac{\Gamma'(1-s)}{\Gamma(1-s)} + \frac{2}{\pi}\cot(\pi s/2). \tag{10}$$

Now $\Re(1-s) = 1-\sigma \ge 2$, so that as before

$$\left|\frac{\zeta'(1-s)}{\zeta(1-s)}\right| \ll 1. \tag{11}$$

By Lemma 2 of the handout on Stirling's Formula (note that the argument of $1-s$ is between $-\pi/2$ and $\pi/2$)

$$\left|\frac{\Gamma'(1-s)}{\Gamma(1-s)}\right| = \log(1-s) + O(1/|s-1|) \ll \log(|s|+4). \tag{12}$$

Finally, since $\pi s/2$ is bounded away from the poles of the cotangent function by hypothesis

$$|\cot(\pi s/2)| = \left|i + \frac{2i}{e^{i\pi s}-1}\right| \ll 1. \tag{13}$$

Proposition 2 follows from (10)-(13).