Math 680 Fall 2020

The Distribution of the Zeros of Xi

As a final result, we will prove the following asymptotic result on the distribution of the non-trivial zeros of the zeta function (= the zeros of the xi function).

**Theorem**: For $T$ sufficiently large

$$N(T) = \frac{T}{2\pi} \log \left( \frac{T}{2\pi} \right) - \frac{T}{2\pi} + O(\log T).$$

**Corollary** (Theorem 2 of the handout on the xi function): For all positive $T$

$$N(T + 1) - N(T) \ll \log(T + 1).$$

Note that the above referenced Theorem 2 implies that $N(T) \ll T \log(T + 2)$. Obviously the Theorem above is a significant sharpening of that estimate.

Proof: Given the error term in the Theorem, there is no harm in “adjusting” the parameter $T$ slightly so as to avoid any zeros, i.e., we may assume that $\Im(\rho) \neq T$ for all zeros $\rho$. Then $N(T)$ is exactly the number of zeros inside the rectangular contour $R$ with vertices $2$, $2 + iT$, $-1 + iT$, and $-1$. In particular, by the argument principle $2\pi N(T)$ is precisely the change in the argument of $\xi(s)$ along the contour $R$. Denote the change in the argument along any contour $C$ by $\Delta_C \text{Arg}$.

Via the representation

$$\zeta(s) = \frac{1}{1 - 2^{1 - s}} \sum_{n \geq 1} \frac{(-1)^n}{n^s},$$

valid whenever $\sigma > 0$ and $s \neq 1$, we easily see that $\zeta(\sigma) < 0$ when $0 < \sigma < 1$. Of course, we already knew that $\zeta(\sigma) > 0$ when $\sigma > 1$. By the infinite product definition of the Gamma function we see that $\Gamma(\sigma/2) > 0$ whenever $\sigma > 0$, and $\pi^{-\sigma/2} > 0$, too. Thus by the definition of the xi function we conclude that $\xi(\sigma) > 0$ whenever $\sigma > 0$. But then the functional equation for $\xi(s)$ implies that $\xi(\sigma) > 0$ for all real $\sigma$. In particular, the argument of $\xi(s)$ does not change along the lower edge of our rectangular contour $R$. Via the functional equation for $\xi(s)$ we conclude that

$$\pi N(T) = (1/2) \Delta_R \text{Arg} \xi(s) = \Delta_L \text{Arg} \xi(s),$$

where $L$ is the contour consisting of the vertical line segment from $2$ up to $2 + iT$ and then from there back (to the left) to $1/2 + iT$

From the definitions we have

$$\Delta_L \text{Arg} \xi(s) = \Delta_L \text{Arg} ((1/2)s(s - 1)) + \Delta_L \text{Arg} \zeta(s) + \Delta_L \text{Arg} \Gamma(s/2) + \Delta_L \text{Arg} \pi^{-s/2}. \tag{2}$$
For the first summand in (2) we compute

\[ \Delta_L \text{Arg}(1/2)s(s - 1) = \text{Arg}((1/2)(1/2 + iT)(iT - 1/2)) - \text{Arg}1 \]

(3)

\[ = \text{Arg}(- (1/2)|1/2 + iT|^2) - 0 \]

\[ = \pi. \]

Similarly, since \( \pi^{-s/2} = \pi^{-1} \) and \( \Gamma(s/2) = \Gamma(1) = 1 \) (both positive real numbers) at \( s = 2 \),

(4)

\[ \Delta_L \text{Arg}\pi^{-s/2} = \text{Arg}\pi^{-1/2-i(T/2)} = -\frac{T}{2} \log \pi \]

and

(5)

\[ \Delta_L \text{Arg}(\Gamma(s/2)) = \text{Arg}\Gamma(1/4 + iT/2). \]

Now since we’re avoiding the negative real axis here, Stirling’s Formula applies giving

\[ \log \Gamma(s) = \log(\sqrt{2\pi}) + (s - 1/2) \log s - s + \log(1 + O(1/|s|)) \]

\[ = \log(\sqrt{2\pi}) + (s - 1/2) \log s - s + O(1/|s|). \]

Using this yields

\[ \text{Arg}\Gamma(1/4 + iT/2) = 0 + (T/2) \log |1/4 + iT/2| - (1/4) \text{Arg}(1/4 + iT/2) - T/2 + O(1) \]

(6)

\[ = (T/2)(\log(T/2) + \log |i + 1/2T|) - T/2 + O(1) \]

\[ = (T/2) \log(T/2) - T/2 + O(1). \]

Hence by (2)-(6)

(7)

\[ \Delta_L \text{Arg}\zeta(s) = \frac{T}{2} \log(T/2\pi) - \frac{T}{2} + \Delta_L \text{Arg}\zeta(s) + O(1). \]

It remains to estimate \( \Delta_L \text{Arg}\zeta(s) \). We denote the first (vertical) line segment of \( L \) by \( L_1 \) and the second (horizontal) segment by \( L_2 \); we have

(8)

\[ \Delta_L \text{Arg}\zeta(s) = \Delta_{L_1} \text{Arg}\zeta(s) + \Delta_{L_2} \text{Arg}\zeta(s). \]

Once more, on \( L_1 \) we’re starting at a positive real number \( \zeta(2) \) so that \( \Delta_{L_1} \text{Arg}\zeta(s) = \text{Arg}\zeta(2 + iT) \). To estimate this quantity, we first recall that whenever \( \sigma > 1 \)

\[ \frac{d \log \zeta(s)}{ds} = \frac{\zeta'(s)}{\zeta(s)} = -\sum_{n \geq 1} \frac{\Lambda(n)}{n^\sigma}. \]

Moreover, the sum is absolutely convergent, so that we may integrate term-by-term to get

\[ \log \zeta(s) = \sum_{n \geq 1} \frac{\Lambda(n)}{n^\sigma \log n}. \]
(We’ve done this before.) Obviously we may apply this to the case \( s = 2 + iT \) to get

\[
|\log \zeta(2 + iT)| = \left| \sum_{n \geq 1} \frac{\Lambda(n)}{n^{2+iT} \log n} \right| \\
\leq \sum_{n \geq 1} \frac{1}{n^2} \\
\ll 1.
\]

But in general \( \log z = \log |z| + i \text{Arg} z \). In particular, \( |\text{Arg} z| \leq |\log z| \) for all \( z \). We thus see that

\[
(9) \quad \Delta L_1 \text{Arg} \zeta(s) = O(1).
\]

Turning to the second piece of \( L \),

\[
\Delta L_2 \text{Arg} \zeta(s) = \text{Arg} \zeta(1/2 + iT) - \text{Arg} \zeta(2 + iT) \\
= \Im \log \zeta(1/2 + iT) - \Im \log \zeta(2 + iT) \\
= \Im \left( \log \zeta(s) \right|_{s=2}^{1/2} \\
= \int_{1/2}^{1/2} \Im \left( \frac{\zeta'(\sigma + iT)}{\zeta(\sigma + iT)} \right) d\sigma.
\]

We now appeal to Theorem 3 of the handout on the xi function (assuming \( T \) is sufficiently large):

\[
\frac{\zeta'(\sigma + iT)}{\zeta(\sigma + iT)} = \frac{-1}{\sigma + iT - 1} + \sum_{\rho \atop |\Im(\rho) - T| \leq 1} \frac{1}{\sigma + iT - \rho} + O(\log T).
\]

Taking imaginary parts yields (writing \( \rho = \beta + it \))

\[
\Im \left( \frac{\zeta'(\sigma + iT)}{\zeta(\sigma + iT)} \right) = \sum_{\rho \atop |\Im(\rho) - T| \leq 1} \Im \left( \frac{1}{\sigma + iT - \rho} \right) + O(\log T) \\
= \sum_{|t-T| \leq 1} \Im \left( \frac{1}{\sigma - \beta + iT - t} \right) + O(\log T) \\
= \sum_{|t-T| \leq 1} \frac{T - t}{(\sigma - \beta)^2 + (T - t)^2} + O(\log T).
\]

Now by this and (10)

\[
|\Delta L_2 \text{Arg} \zeta(s)| \leq \sum_{\rho \atop |t-T| \leq 1} \int_{1/2}^{2} \frac{|T - t|}{(\sigma - \beta)^2 + (T - t)^2} d\sigma + O(\log T) \\
\leq \sum_{\rho \atop |t-T| \leq 1} \int_{1/2}^{2} \frac{1}{(\sigma - \beta)^2 + 1} d\sigma + O(\log T) \\
< \sum_{\rho \atop |t-T| \leq 1} \int_{-\infty}^{\infty} \frac{1}{x^2 + 1} dx + O(\log T) \\
\ll \sum_{\rho \atop |\Im(\rho) - T| \leq 1} 1 + O(\log T).
\]
But by Theorem 2 of the xi handout

\[ \sum_{\rho \mid \Im(\rho) \leq |T - T|} 1 = N(T + 1) - N(T) + N(T) - N(T - 1) \ll \log T \]

for \( T \) sufficiently large, so that

(11) \[ \Delta_{L_2} \text{Arg}\zeta(s) = O(\log T). \]

The Theorem follows from (1), (7), (8), (9), and (11).