Math 680 Fall 2017

The Distribution of the Zeros of Xi

As a final result, we will prove the following asymptotic result on the distribution of the non-trivial zeros of the zeta function (= the zeros of the xi function).

Theorem : For T sufficiently large

$$N(T) = \frac{T}{2\pi}\log\left(\frac{T}{2\pi}\right) - \frac{T}{2\pi} + O(\log T).$$

Corollary (Theorem 2 of the handout on the xi function): For all positive T

$$N(T+1) - N(T) \ll \log(T+2).$$

Note that the above referenced Theorem 2 implies that $N(T) \ll T\log(T+2)$. Obviously the Theorem above is a significant sharpening of that estimate.

Proof: Given the error term in the Theorem, there is no harm in "adjusting" the parameter T slightly so as to avoid any zeros, i.e., we may assume that $\Im(\rho) \neq T$ for all zeros ρ. Then $N(T)$ is exactly the number of zeros inside the rectangular contour R with vertices 2, $2+iT$, $-1+iT$, and -1. In particular, by the argument principle $2\pi N(T)$ is precisely the change in the argument of $\xi(s)$ along the contour R. Denote the change in the argument along any contour C by $\Delta_C\mathrm{Arg}$.

Via the representation

$$\zeta(s) = \frac{1}{1-2^{1-s}}\sum_{n\geq 1}\frac{(-1)^{n+1}}{n^s},$$

valid whenever $\sigma > 0$ and $s \neq 1$, we easily see that $\zeta(\sigma) < 0$ when $0 < \sigma < 1$. Of course, we already knew that $\zeta(\sigma) > 0$ when $\sigma > 1$. By the infinite product definition of the Gamma function we see that $\Gamma(\sigma/2) > 0$ whenever $\sigma > 0$, and $\pi^{-\sigma/2} > 0$, too. Thus by the definition of the xi function we conclude that $\xi(\sigma) > 0$ whenever $\sigma > 0$. But then the functional equation for $\xi(s)$ implies that $\xi(\sigma) > 0$ for *all* real σ. In particular, the argument of $\xi(s)$ does not change along the lower edge of our rectangular contour R. Via the functional equation for $\xi(s)$ we conclude that

(1) $$\pi N(T) = (1/2)\Delta_R\mathrm{Arg}\xi(s) = \Delta_L\mathrm{Arg}\xi(s),$$

where L is the contour consisting of the vertical line segment from 2 up to $2+iT$ and then from there back (to the left) to $1/2+iT$

From the definitions we have

(2) $$\Delta_L\mathrm{Arg}\xi(s) = \Delta_L\mathrm{Arg}\big((1/2)s(s-1)\big) + \Delta_L\mathrm{Arg}\zeta(s) + \Delta_L\mathrm{Arg}\Gamma(s/2) + \Delta_L\mathrm{Arg}\pi^{-s/2}.$$

For the first summand in (2) we compute

$$\begin{aligned}\Delta_L \mathrm{Arg}\big((1/2)s(s-1)\big) &= \mathrm{Arg}\big((1/2)(1/2+iT)(iT-1/2)\big) - \mathrm{Arg}1 \\ &= \mathrm{Arg}\big(-(1/2)|1/2+iT|^2\big) - 0 \\ &= \pi.\end{aligned} \tag{3}$$

Similarly, since $\pi^{-s/2} = \pi^{-1}$ and $\Gamma(s/2) = \Gamma(1) = 1$ (both positive real numbers) at $s = 2$,

$$\Delta_L \mathrm{Arg}\pi^{-s/2} = \mathrm{Arg}\pi^{-(1/2)-i(T/2)} = \frac{-T}{2}\log\pi \tag{4}$$

and

$$\Delta_L \mathrm{Arg}\big(\Gamma(s/2)\big) = \mathrm{Arg}\Gamma(1/4+iT/2)\big). \tag{5}$$

Now since we're avoiding the negative real axis here, Stirling's Formula applies giving

$$\begin{aligned}\log\Gamma(s) &= \log(\sqrt{2\pi}) + (s-1/2)\log s - s + \log\big(1+O(1/|s|)\big) \\ &= \log(\sqrt{2\pi}) + (s-1/2)\log s - s + O(1/|s|).\end{aligned}$$

Using this yields

$$\begin{aligned}\mathrm{Arg}\Gamma(1/4+iT/2) &= 0 + (T/2)\log|1/4+iT/2| - (1/4)\mathrm{Arg}(1/4+iT/2) - T/2 + O(1) \\ &= (T/2)\big(\log(T/2) + \log|i+1/2T|\big) - T/2 + O(1) \\ &= (T/2)\log(T/2) - T/2 + O(1).\end{aligned} \tag{6}$$

Hence by (2)-(6)

$$\Delta_L \mathrm{Arg}\xi(s) = \frac{T}{2}\log(T/2\pi) - \frac{T}{2} + \Delta_L \mathrm{Arg}\zeta(s) + O(1). \tag{7}$$

It remains to estimate $\Delta_L \mathrm{Arg}\zeta(s)$. We denote the first (vertical) line segment of L by L_1 and the second (horizontal) segment by L_2; we have

$$\Delta_L \mathrm{Arg}\zeta(s) = \Delta_{L_1} \mathrm{Arg}\zeta(s) + \Delta_{L_2} \mathrm{Arg}\zeta(s). \tag{8}$$

Once more, on L_1 we're starting at a positive real number $\zeta(2)$ so that $\Delta_{L_1} \mathrm{Arg}\zeta(s) = \mathrm{Arg}\zeta(2+iT)$. To estimate this quantity, we first recall that whenever $\sigma > 1$

$$\frac{d\log\zeta(s)}{ds} = \frac{\zeta'(s)}{\zeta(s)} = -\sum_{n\geq 1}\frac{\Lambda(n)}{n^s}.$$

Moreover, the sum is absolutely convergent, so that we may integrate term-by-term to get

$$\log\zeta(s) = \sum_{n\geq 1}\frac{\Lambda(n)}{n^s \log n}.$$

(We've done this before.) Obviously we may apply this to the case $s = 2 + iT$ to get

$$\begin{aligned}
|\log \zeta(2+iT)| &= \left| \sum_{n \geq 1} \frac{\Lambda(n)}{n^{2+iT} \log n} \right| \\
&\leq \sum_{n \geq 1} \frac{1}{n^2} \\
&\ll 1.
\end{aligned}$$

But in general $\log z = \log |z| + i\text{Arg}z$. In particular, $|\text{Arg}z| \leq |\log z|$ for all z. We thus see that

$$\Delta_{L_1} \text{Arg}\zeta(s) = O(1). \tag{9}$$

Turning to the second piece of L,

$$\begin{aligned}
\Delta_{L_2} \text{Arg}\zeta(s) &= \text{Arg}\zeta(1/2 + iT) - \text{Arg}\zeta(2+iT) \\
&= \Im \log \zeta(1/2+iT) - \Im \log \zeta(2+iT) \\
&= \Im \left(\quad \log \zeta(s) \Big|_{s=2}^{1/2} \right) \\
&= \int_2^{1/2} \Im \left(\frac{\zeta'(\sigma+iT)}{\zeta(\sigma+iT)} \right) d\sigma.
\end{aligned} \tag{10}$$

We now appeal to Theorem 3 of the handout on the xi function (assuming T is sufficiently large):

$$\frac{\zeta'(\sigma+iT)}{\zeta(\sigma+iT)} = \frac{-1}{\sigma+iT-1} + \sum_{\substack{\rho \\ |\Im(\rho)-T| \leq 1}} \frac{1}{\sigma+iT-\rho} + O(\log T).$$

Taking imaginary parts yields (writing $\rho = \beta + it$)

$$\begin{aligned}
\Im \left(\frac{\zeta'(\sigma+iT)}{\zeta(\sigma+iT)} \right) &= \sum_{\substack{\rho \\ |\Im(\rho)-T| \leq 1}} \Im \left(\frac{1}{\sigma+iT-\rho} \right) + O(\log T) \\
&= \sum_{\substack{\rho \\ |t-T| \leq 1}} \Im \left(\frac{1}{\sigma-\beta+i(T-t)} \right) + O(\log T) \\
&= \sum_{\substack{\rho \\ |t-T| \leq 1}} \frac{T-t}{(\sigma-\beta)^2+(T-t)^2} + O(\log T).
\end{aligned}$$

Now by this and (10)

$$\begin{aligned}
|\Delta_{L_2} \text{Arg}\zeta(s)| &\leq \sum_{\substack{\rho \\ |t-T| \leq 1}} \int_{1/2}^{2} \frac{|T-t|}{(\sigma-\beta)^2+(T-t)^2} \, d\sigma + O(\log T) \\
&\leq \sum_{\substack{\rho \\ |t-T| \leq 1}} \int_{1/2}^{2} \frac{1}{(\sigma-\beta)^2+1} \, d\sigma + O(\log T) \\
&< \sum_{\substack{\rho \\ |t-T| \leq 1}} \int_{-\infty}^{\infty} \frac{1}{x^2+1} \, dx + O(\log T) \\
&\ll \sum_{\substack{\rho \\ |\Im(\rho)-T| \leq 1}} 1 + O(\log T).
\end{aligned}$$

But by Theorem 2 of the xi handout

$$\sum_{\substack{\rho \\ |\Im(\rho)-T|\leq 1}} 1 = N(T+1) - N(T) + N(T) - N(T-1) \ll \log T$$

for T sufficiently large, so that

$$\Delta_{L_2} \text{Arg}\zeta(s) = O(\log T). \tag{11}$$

The Theorem follows from (1), (7), (8), (9), and (11).