Theorem (1)

Let $O_S$ be a ring of $S$-integers of $K$ and let $O_S'$ denote the integral closure of $O_S$ in $F$.

For any basis $\{\alpha_1, \ldots, \alpha_n\}$ of $F$ over $K$ there are non-zero $a_i \in O_S$, $i = 1, \ldots, n$ such that $a_i \alpha_i \in O_S'$ for all $i$.

Thus there is a basis for $F$ contained in $O_S'$.

If $\{\alpha_1, \ldots, \alpha_n\} \subset O_S'$ is a basis for $F$ over $K$ and $\{\alpha^*_1, \ldots, \alpha^*_n\}$ is the dual basis, then

$$\sum_{i=1}^n \alpha_i O_S \subseteq O_S' \subseteq \sum_{i=1}^n \alpha^*_i O_S.$$
Let $O_S$ be a ring of $S$-integers of $K$ and let $O'_S$ denote the integral closure of $O_S$ in $F$. For any basis \( \{\alpha_1, \ldots, \alpha_n\} \) of $F$ over $K$, there are non-zero $a_i \in O_S$, $i = 1, \ldots, n$ such that $a_i \alpha_i \in O'_S$ for all $i$. Thus there is a basis for $F$ contained in $O'_S$. If \( \{\alpha_1, \ldots, \alpha_n\} \subset O'_S \) is a basis for $F$ over $K$ and \( \{\alpha^*_1, \ldots, \alpha^*_n\} \) is the dual basis, then

\[
\sum_{i=1}^n \alpha_i O_S \subseteq O'_S \subseteq \sum_{i=1}^n \alpha^*_i O_S.
\]

If $O_S$ is a principal ideal domain, then there is a basis \( \{\alpha_1, \ldots, \alpha_n\} \) such that $O'_S = \sum_{i=1}^n \alpha_i O_S$. 
Theorem (1)

Let $\mathcal{O}_S$ be a ring of $S$-integers of $K$ and let $\mathcal{O}'_S$ denote the integral closure of $\mathcal{O}_S$ in $F$. For any basis $\{\alpha_1, \ldots, \alpha_n\}$ of $F$ over $K$, there are non-zero $a_i \in \mathcal{O}_S$, $i = 1, \ldots, n$ such that $a_i \alpha_i \in \mathcal{O}'_S$ for all $i$. Thus there is a basis for $F$ contained in $\mathcal{O}'_S$. If $\{\alpha_1, \ldots, \alpha_n\} \subset \mathcal{O}'_S$ is a basis for $F$ over $K$ and $\{\alpha^*_1, \ldots, \alpha^*_n\}$ is the dual basis, then $\sum_{i=1}^n \alpha_i \mathcal{O}_S \subseteq \mathcal{O}'_S \subseteq \sum_{i=1}^n \alpha^*_i \mathcal{O}_S$. If $\mathcal{O}_S$ is a principal ideal domain, then there is a basis $\{\alpha_1, \ldots, \alpha_n\}$ such that $\mathcal{O}'_S = \sum_{i=1}^n \alpha_i \mathcal{O}_S$. 

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Integral Closure and Complementary Modules, III

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$$\mathcal{O}'_S = \sum_{i=1}^n \alpha_i \mathcal{O}_S.$$
Corollary

Let \( w \in M(K) \) be a non-archimedean place. Then the integral closure \( O_w' \) of \( O_w \) in \( F \) is

\[
O_w' = \bigcap_{v \in M(F)} v | w O_v.
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There is a basis \( \{ \alpha_1, \ldots, \alpha_n \} \) of \( F \) over \( K \) such that

\[
O_w' = \sum_{i=1}^{n} \alpha_i O_w.
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The basis asserted to exist in the Corollary above is called a local integral basis (with respect to the place \( w \)).

Theorem (2)

Any basis \( \{ \alpha_1, \ldots, \alpha_n \} \) of \( F \) over \( K \) is a local integral basis for almost all places \( w \in M(K) \) (i.e., for all but finitely many places).
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Let $w \in M(K)$ be a non-archimedean place.

The integral closure $O'_w$ of $O_w$ in $F$ is

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Any basis \( \{\alpha_1, \ldots, \alpha_n\} \) of \( F \) over \( K \) is a local integral basis for almost all places \( w \in M(K) \) (i.e., for all but finitely many places).
Proof:

Consider the dual basis \{α^∗_1, ..., α^∗_n\}.

We have a total of 2^n elements of \(F\) as follows: α_1, ..., α_n, α^∗_1, ..., α^∗_n each with at most \(n\) non-zero coefficients (besides 1) in \(K\) for their respective minimal polynomials.

Each of these coefficients has negative order at finitely many non-archimedean places; let \(S\) denote the set of all these places, which is necessarily finite.

Now by construction all α_i and α^∗_j are elements of \(O'_{w}\) whenever \(w \not\in S\).

Therefore by Theorem 1 (twice)

\[
\sum_{i=1}^{n} α_i O_w \subseteq O'_{w} \subseteq \sum_{i=1}^{n} α^∗_i O_w \subseteq O'_{w} \subseteq \sum_{i=1}^{n} α_i O_w
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(note that \{α_1, ..., α_n\} is the dual basis of \{α^∗_1, ..., α^∗_n\}).

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Another application of localizing is the following.
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**Theorem (3)**

If \( \{1, \alpha, \ldots, \alpha^{n-1}\} \) is a local integral basis for a number field \( K \) with respect to a prime \( p \), then the conclusion of Dedekind's Theorem is valid for the prime \( p \). Specifically, in the statement of Dedekind's Theorem \( \mathbb{Z} \) may be replaced by any Dedekind domain, \( \mathbb{Q} \) by its quotient field, \( \mathcal{O}_K \) by its integral closure in any (finite) extension of the quotient field, and \( p \) by a prime of the Dedekind domain. The proof of Theorem 3 is the same as the proof of Dedekind's Theorem with just the obvious changes (essentially just replacing words as we did in the statement above).
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The proof of Theorem 3 is the same as the proof of Dedekind’s Theorem with just the obvious changes (essentially just replacing words as we did in the statement above).
Proposition

With the notation above, the complementary module $C_w := \{ \alpha \in F : \text{Tr}_{F/K}(\alpha \beta) \in O_w \forall \beta \in O'_w \}$ is an $O'_w$-module containing $O'_w$ and $C_w = O'_w$ for almost all places $w \in M(K)$.

Further, if $\{\alpha_1, ..., \alpha_n\}$ is an integral basis for $O'_w$ over $O_w$, then $C_w = \sum_{i=1}^{n} \alpha_i^* O_w$, where $\{\alpha_1^*, ..., \alpha_n^*\}$ is the dual basis.

There is an element $\pi_w \in F$ such that $C_w = \pi_w O'_w$, $\text{ord}_v(\pi_w) \leq 0$ for all $v \in M(F)$, $v | w$, and $C_w = \pi O'_w$ if and only if $\text{ord}_v(\pi) = \text{ord}_v(\pi_w)$ for all $v | w$.

Finally, $C_w = O'_w$ for almost all $w \in M(K)$. 
Proposition

With the notation above, the complementary module

\[ C_w := \{ \alpha \in F : \text{Tr}_{F/K}(\alpha \beta) \in \mathfrak{O}_w \text{ all } \beta \in \mathfrak{O}'_w \} \]

is an \( \mathfrak{O}'_w \)-module containing \( \mathfrak{O}'_w \) and \( C_w = \mathfrak{O}'_w \) for almost all places \( w \in M(K) \).
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With the notation above, the complementary module

$$C_w := \{ \alpha \in F : \text{Tr}_{F/K}(\alpha \beta) \in \mathcal{O}_w \text{ all } \beta \in \mathcal{O}'_w \}$$

is an $\mathcal{O}'_w$-module containing $\mathcal{O}'_w$ and $C_w = \mathcal{O}'_w$ for almost all places $w \in M(K)$. Further, if $\{\alpha_1, \ldots, \alpha_n\}$ is an integral basis for $\mathcal{O}'_w$ over $\mathcal{O}_w$, then

$$C_w = \sum_{i=1}^{n} \alpha_i^* \mathcal{O}_w,$$

where $\{\alpha_1^*, \ldots, \alpha_n^*\}$ is the dual basis.

There is an element $\pi_w \in F$ such that $C_w = \pi_w \mathcal{O}'_w$, $\text{ord}_v(\pi_w) \leq 0$ for all $v \in M(F)$, $v | w$, and $C_w = \pi \mathcal{O}'_w$ if and only if $\text{ord}_v(\pi) = \text{ord}_v(\pi_w)$ for all $v | w$.

Finally, $C_w = \mathcal{O}'_w$ for almost all $w \in M(K)$. 
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With the notation above, the complementary module

\[ C_w := \{ \alpha \in F : \text{Tr}_{F/K}(\alpha \beta) \in \mathcal{O}_w \text{ all } \beta \in \mathcal{O}_w' \} \]

is an \( \mathcal{O}_w' \)-module containing \( \mathcal{O}_w' \) and \( C_w = \mathcal{O}_w' \) for almost all places \( w \in M(K) \). Further, if \( \{\alpha_1, \ldots, \alpha_n\} \) is an integral basis for \( \mathcal{O}_w' \) over \( \mathcal{O}_w \), then

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With the notation above, the complementary module

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Finally, \( C_w = \mathcal{O}'_w \) for almost all \( w \in M(K) \).
Proof:

Lemma 4 from last Friday shows that $C_w$ is an $O'_w$-module containing $O'_w$.

Suppose $\alpha \in C_w$ and write $\alpha = \sum_{i=1}^{n} a_i \alpha^*_i$ with $a_i \in K$ for all $i$.

Since $\text{Tr}_{F/K}(\alpha^*_j) \in O_w$ for all $j$, by properties of the dual basis we see that $a_i \in O_w$ for all $i$.

On the other hand, if $\alpha \in \sum_{i=1}^{n} \alpha^*_i O_w$ and $\beta \in O'_w$, then writing $\beta = \sum_{j=1}^{n} b_j \alpha_j$ with $b_j \in O_w$ for all $j$ and taking the trace yields $\alpha \in C_w$.

Therefore $C_w = \sum_{i=1}^{n} \alpha^*_i O_w$.

That $C_w = O'_w$ for almost all places follows from Theorem 2 above (both $\{\alpha_1, ..., \alpha_n\}$ and $\{\alpha^*_1, ..., \alpha^*_n\}$ are local integral bases for almost all $w$).
Proof: Lemma 4 from last Friday shows that $C_w$ is an $\mathcal{O}'_w$-module containing $\mathcal{O}'_w$. Suppose $\alpha \in C_w$ and write $\alpha = \sum_{i=1}^{n} a_i \alpha^*_i$ with $a_i \in K$ for all $i$. Since $\text{Tr} F/K(\alpha_j) \in \mathcal{O}_w$ for all $j$, by properties of the dual basis we see that $a_i \in \mathcal{O}_w$ for all $i$. On the other hand, if $\alpha \in \sum_{i=1}^{n} \alpha^*_i \mathcal{O}_w$ and $\beta \in \mathcal{O}'_w$, then writing $\beta = \sum_{j=1}^{n} b_j \alpha_j$ with $b_j \in \mathcal{O}_w$ for all $j$ and taking the trace yields $\alpha \in C_w$. Therefore $C_w = \sum_{i=1}^{n} \alpha^*_i \mathcal{O}_w$. That $C_w = \mathcal{O}'_w$ for almost all places follows from Theorem 2 above (both $\{\alpha_1, \ldots, \alpha_n\}$ and $\{\alpha^*_1, \ldots, \alpha^*_n\}$ are local integral bases for almost all $w$).
**Proof:** Lemma 4 from last Friday shows that $\mathcal{C}_w$ is an $\mathcal{O}'_w$-module containing $\mathcal{O}'_w$.

Suppose $\alpha \in \mathcal{C}_w$ and write $\alpha = \sum_{i=1}^{n} a_i \alpha_i^*$ with $a_i \in K$ for all $i$. 
**Proof:** Lemma 4 from last Friday shows that $C_w$ is an $\mathcal{O}'_w$-module containing $\mathcal{O}'_w$.

Suppose $\alpha \in C_w$ and write $\alpha = \sum_{i=1}^{n} a_i \alpha^*_i$ with $a_i \in K$ for all $i$. Since $\text{Tr}_{F/K}(\alpha \alpha_j) \in \mathcal{O}_w$ for all $j$, $C_w = \sum_{i=1}^{n} \alpha^*_i \mathcal{O}_w$.

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**Proof:** Lemma 4 from last Friday shows that $C_w$ is an $\mathcal{O}_w'$-module containing $\mathcal{O}_w'$. 

Suppose $\alpha \in C_w$ and write $\alpha = \sum_{i=1}^{n} a_i \alpha_i^*$ with $a_i \in K$ for all $i$. Since $\text{Tr}_{F/K}(\alpha \alpha_j) \in \mathcal{O}_w$ for all $j$, by properties of the dual basis we see that $a_i \in \mathcal{O}_w$ for all $i$.

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Suppose $\alpha \in C_w$ and write $\alpha = \sum_{i=1}^{n} a_i \alpha_i^*$ with $a_i \in K$ for all $i$. Since $\text{Tr}_{F/K}(\alpha \alpha_j) \in \mathcal{O}_w$ for all $j$, by properties of the dual basis we see that $a_i \in \mathcal{O}_w$ for all $i$.

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Therefore $C_w = \sum_{i=1}^{n} \alpha_i^* \mathcal{O}_w$.

That $C_w = \mathcal{O}'_w$ for almost all places follows from Theorem 2 above (both $\{\alpha_1, \ldots, \alpha_n\}$ and $\{\alpha_1^*, \ldots, \alpha_n^*\}$ are local integral bases for almost all $w$).
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Now write \( C_w = \sum_{i=1}^{n} \delta_i \mathfrak{o}_w \) with \( \delta_i \in F \) for all \( i \) and choose a \( d \in K \) with \( \text{ord}_w(d) \geq 0 \) and \( \text{ord}_w(d) \geq -\text{ord}_v(\delta_i) \) for all \( i \) and all \( v \mid w \).
Now write $C_w = \sum_{i=1}^{n} \delta_i \mathcal{O}_w$ with $\delta_i \in F$ for all $i$ and choose a $d \in K$ with $\text{ord}_w(d) \geq 0$ and $\text{ord}_w(d) \geq -\text{ord}_v(\delta_i)$ for all $i$ and all $v | w$. We then have

$$\text{ord}_v(d\delta_i) = e_{v|w} \text{ord}_w(d) + \text{ord}_v(\delta_i) \geq 0$$
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It is clear that \( dC_w \) is an ideal of the ring \( \mathfrak{O}_w' \), which is a principal ideal domain by Lemma 3 from last Friday, so that \( dC_w = \delta \mathfrak{O}_w' \).
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It is clear that $dC_w$ is an ideal of the ring $\mathcal{O}'_w$, which is a principal ideal domain by Lemma 3 from last Friday, so that $dC_w = \delta \mathcal{O}'_w$.

Set $\pi_w = \delta/d$. 
Now write $C_w = \sum_{i=1}^{n} \delta_i \mathcal{O}_w$ with $\delta_i \in F$ for all $i$ and choose a $d \in K$ with $\text{ord}_w(d) \geq 0$ and $\text{ord}_w(d) \geq -\text{ord}_v(\delta_i)$ for all $i$ and all $v | w$. We then have

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Set $\pi_w = \delta/d$. Then $\text{ord}_v(\pi_w) \leq 0$ for all places $v | w$ since $\mathcal{O}'_w \subseteq C_w$. 
Now write $C_w = \sum_{i=1}^n \delta_i \mathcal{O}_w$ with $\delta_i \in F$ for all $i$ and choose a $d \in K$ with $\text{ord}_w(d) \geq 0$ and $\text{ord}_w(d) \geq -\text{ord}_v(\delta_i)$ for all $i$ and all $v|w$. We then have

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Set $\pi_w = \delta/d$. Then $\text{ord}_v(\pi_w) \leq 0$ for all places $v|w$ since $\mathcal{O}'_w \subseteq C_w$. Further, $\pi_w \mathcal{O}'_w = \pi'_w \mathcal{O}'_w$ if and only if $\pi_w/\pi'_w, \pi'_w/\pi_w \in \mathcal{O}'_w$. 
Now write $C_w = \sum_{i=1}^{n} \delta_i O_w$ with $\delta_i \in F$ for all $i$ and choose a $d \in K$ with $\text{ord}_w(d) \geq 0$ and $\text{ord}_w(d) \geq -\text{ord}_v(\delta_i)$ for all $i$ and all $v \mid w$. We then have

$$\text{ord}_v(d\delta_i) = e_{v \mid w} \text{ord}_w(d) + \text{ord}_v(\delta_i) \geq 0$$

so that by the Corollary to Theorem 1 $dC_w \subseteq O'_w$.

It is clear that $dC_w$ is an ideal of the ring $O'_w$, which is a principal ideal domain by Lemma 3 from last Friday, so that $dC_w = \delta O'_w$.

Set $\pi_w = \delta / d$. Then $\text{ord}_v(\pi_w) \leq 0$ for all places $v \mid w$ since $O'_w \subseteq C_w$. Further, $\pi_w O'_w = \pi'_w O'_w$ if and only if $\pi_w / \pi'_w, \pi'_w / \pi_w \in O'_w$, which is the case if and only if $\text{ord}_v(\pi_w) = \text{ord}_v(\pi'_w)$ for all $v \mid w$. 
The Different

The Proposition is very reminiscent of our discussion about the different of a number field from February 17.

With the definitions and notation above, we may restate things as follows.

Definition

For a number field \( K \) and \( \mathbb{Z} \)-basis \( \alpha_1, \ldots, \alpha_n \) of \( \mathcal{O}_K \), the different \( D_K \) is the ideal with \( \mathbb{Z} \)-basis \( \alpha^*_1, \ldots, \alpha^*_n \).

We now extend this notion.

Definition

Returning to the previous situation where \( F \) is a separable extension of \( K \), let \( w \in M(K) \) be a non-archimedean place and \( v \in M(F) \) lie above \( w \).

The different exponent \( d_{v|w} \) is the non-negative integer given by

\[
d_{v|w} = -\text{ord}_v(\pi_w)
\]

where \( \pi_w \in F \) is a generator for the complementary module:

\[
C_w = \pi_w \mathcal{O}'_w.
\]

The local different \( D_{v|w} \) is the ideal \( P_{d_{v|w}} \) in the number field case and the divisor \( d_{v|w} \cdot v \) in the function field case.
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Returning to the previous situation where \( F \) is a separable extension of \( K \), let \( w \in \mathcal{M}(K) \) be a non-archimedean place and \( v \in \mathcal{M}(F) \) lie above \( w \).

The different exponent \( d_{v|w} \) is the non-negative integer given by

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Returning to the previous situation where $F$ is a separable extension of $K$, let $w \in M(K)$ be a non-archimedean place and $v \in M(F)$ lie above $w$. The different exponent $d_v|w$ is the non-negative integer given by $d_v|w = -\text{ord}_v(\pi^w)$ where $\pi^w \in F$ is a generator for the complementary module $C_w = \pi^w\mathcal{O}'_w$. The local different $D_v|w$ is the ideal $P_d_v|w v$ in the number field case and the divisor $d_v|w \cdot v$ in the function field case.
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The Proposition is very reminiscent of our discussion about the different of a number field from February 17. With the definitions and notation above, we may restate things as follows.

**Definition**

For a number field $K$ and $\mathbb{Z}$-basis $\alpha_1, \ldots, \alpha_n$ of $\mathcal{O}_K$, the different $D_K$ is the ideal with $\mathbb{Z}$-basis $\alpha^*1, \ldots, \alpha^*n$.

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**Definition**

Returning to the previous situation where $F$ is a separable extension of $K$, let $w \in M(K)$ be a non-archimedean place and $v \in M(F)$ lie above $w$. The different exponent $d_v|w$ is the non-negative integer given by $d_v|w = -\text{ord}_v(\pi_w)$ where $\pi_w \in F$ is a generator for the complementary module $C_w = \pi_w \mathcal{O}_w'$. The local different $D_v|w$ is the ideal $P(d_v|w)v$ in the number field case and the divisor $d_v|w \cdot v$ in the function field case.
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**Definition**

For a number field $K$ and $\mathbb{Z}$-basis $\alpha_1, \ldots, \alpha_n$ of $\mathcal{O}_K$, the *different* $\mathcal{D}_K$ is the ideal with $\mathbb{Z}$-basis $\alpha_1^*, \ldots, \alpha_n^*$. 
The Proposition is very reminiscent of our discussion about the different of a number field from February 17. With the definitions and notation above, we may restate things as follows.

**Definition**

For a number field $K$ and $\mathbb{Z}$-basis $\alpha_1, \ldots, \alpha_n$ of $\mathcal{O}_K$, the *different* $\mathcal{D}_K$ is the ideal with $\mathbb{Z}$-basis $\alpha_1^*, \ldots, \alpha_n^*$.

We now extend this notion.
The Different

The Proposition is very reminiscent of our discussion about the different of a number field from February 17. With the definitions and notation above, we may restate things as follows.

**Definition**

For a number field $K$ and $\mathbb{Z}$-basis $\alpha_1, \ldots, \alpha_n$ of $\mathcal{O}_K$, the different $\mathcal{D}_K$ is the ideal with $\mathbb{Z}$-basis $\alpha_1^*, \ldots, \alpha_n^*$.

We now extend this notion.

**Definition**

Returning to the previous situation where $F$ is a separable extension of $K$, let $w \in M(K)$ be a non-archimedean place and $v \in M(F)$ lie above $w$. The different exponent $d_v|w$ is the non-negative integer given by $d_v|w = -\text{ord}_v(\pi_w)$ where $\pi_w \in F$ is a generator for the complementary module $C_w = \pi_w \mathcal{O}_w'$. The local different $D_v|w$ is the ideal $P_{d_v|w}v$ in the number field case and the divisor $d_v|w \cdot v$ in the function field case.
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\[ C_w = \pi_w \mathcal{O}_w. \]
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Note that we may take $\pi_w = 1$ for almost all places $w$ by Theorem 2.
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**Theorem (Dedekind’s Different Theorem)**
Note that we may take $\pi_w = 1$ for almost all places $w$ by Theorem 2. We thus have the *relative different* $\mathfrak{D}_{F/K}$ which is the product of the local differents in the number field case and the sum of the local differents in the function field case.

**Theorem (Dedekind’s Different Theorem)**

For all non-archimedean places $w \in M(K)$ and all places $v \in M(F)$ lying above $w$ we have $d_{v|w} \geq e_{v|w} - 1$. 
Note that we may take $\pi_w = 1$ for almost all places $w$ by Theorem 2. We thus have the \textit{relative different} $\mathfrak{D}_{F/K}$ which is the product of the local differents in the number field case and the sum of the local differents in the function field case.

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For all non-archimedean places $w \in M(K)$ and all places $v \in M(F)$ lying above $w$ we have $d_{v|w} \geq e_{v|w} - 1$. Further, we have equality if and only if the characteristic of the residue class field doesn’t divide the ramification index $e_{v|w}$. 
Note that we may take $\pi_w = 1$ for almost all places $w$ by Theorem 2. We thus have the *relative different* $\mathfrak{D}_{F/K}$ which is the product of the local differents in the number field case and the sum of the local differents in the function field case.

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*For all non-archimedean places $w \in M(K)$ and all places $v \in M(F)$ lying above $w$ we have $d_v|w \geq e_v|w - 1$. Further, we have equality if and only if the characteristic of the residue class field doesn’t divide the ramification index $e_v|w$.***

**Corollary**
Note that we may take \( \pi_w = 1 \) for almost all places \( w \) by Theorem 2. We thus have the *relative different* \( \mathfrak{D}_{F/K} \) which is the product of the local differents in the number field case and the sum of the local differents in the function field case.

**Theorem (Dedekind’s Different Theorem)**

For all non-archimedean places \( w \in M(K) \) and all places \( v \in M(F) \) lying above \( w \) we have \( d_{v|w} \geq e_{v|w} - 1 \). Further, we have equality if and only if the characteristic of the residue class field doesn’t divide the ramification index \( e_{v|w} \).

**Corollary**

A place \( v \in M(F) \) is ramified if and only if \( \mathfrak{p}_v|\mathfrak{D}_{F/K} \) (as ideals) in the number field case.
Note that we may take $\pi_w = 1$ for almost all places $w$ by Theorem 2. We thus have the *relative different* $\mathfrak{D}_{F/K}$ which is the product of the local differents in the number field case and the sum of the local differents in the function field case.

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**Corollary**

A place $v \in M(F)$ is ramified if and only if $\mathfrak{P}_v | \mathfrak{D}_{F/K}$ (as ideals) in the number field case or $v \leq \mathfrak{D}_{F/K}$ (as divisors) in the function field case.
Note that we may take $\pi_w = 1$ for almost all places $w$ by Theorem 2. We thus have the relative different $\mathfrak{D}_{F/K}$ which is the product of the local different in the number field case and the sum of the local different in the function field case.

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Note that we may take $\pi_w = 1$ for almost all places $w$ by Theorem 2. We thus have the relative different $\mathfrak{D}_{F/K}$ which is the product of the local differents in the number field case and the sum of the local differents in the function field case.

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A place $v \in M(F)$ is ramified if and only if $\mathfrak{P}_v | \mathfrak{D}_{F/K}$ (as ideals) in the number field case or $v \leq \mathfrak{D}_{F/K}$ (as divisors) in the function field case. Almost all places are unramified, and in particular, a given rational prime $p$ is ramified in a number field $K$ if and only if $p$ divides the discriminant.
Theorem

Write \( F = K(\alpha) \) and let \( P(X) = X^n + a_{n-1}X^{n-1} + \cdots + a_0 \in K[X] \) be the minimal polynomial for \( \alpha \).

Suppose \( w \in M(K) \) such that \( a_i \in \mathcal{O}_w \) for all \( i \). Then for all places \( v \in M(F) \) lying over \( w \) we have:

\[ d_v | w \leq \text{ord}_v(P'(\alpha)) ; \{1, \alpha, \ldots, \alpha_{n-1}\} \text{ is a local integral basis at } v \text{ if and only if } d_v | w = \text{ord}_v(P'(\alpha)) \].
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- $d_{v|w} \leq \ord_v \left( P'(\alpha) \right)$;
- $\{1, \alpha, \ldots, \alpha^{n-1}\}$ is a local integral basis at $v$ if and only if $d_{v|w} = \ord_v (P'(\alpha))$. 