Northern Illinois University, Math 681

April 7, 2021
The Strong Approximation Theorem

Recall the following, proven well before the Riemann-Roch Theorem.

Theorem (Weak Approximation Theorem)
Let $K$ be a function field, let $R_1, \ldots, R_n$ be distinct valuation rings of $K$, and denote the corresponding valuations by $v_1, \ldots, v_n$. Let $\alpha_1, \ldots, \alpha_n \in K$ and $z_1, \ldots, z_n \in \mathbb{Z}$. There is an $\alpha \in K$ with $v_i(\alpha - \alpha_i) = z_i$ for all $i = 1, \ldots, n$.

It's now time to improve upon that result.

Theorem (Strong Approximation Theorem)
Let $K$ be a function field and $S \subseteq M(K)$ be a proper and non-empty subset of places of $K$. Let $v_1, \ldots, v_n \in S$ with corresponding $\alpha_1, \ldots, \alpha_n \in K$ and $z_1, \ldots, z_n \in \mathbb{Z}$. Then there is an $\alpha \in K$ such that $\text{ord}_{v_i}(\alpha - \alpha_i) = z_i$ for all $i = 1, \ldots, n$ and $\text{ord}_{v}(\alpha) \geq 0$ for all places $v \in S, v \neq v_1, \ldots, v_n$. 

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\l(W - A) = \dim_{\mathbb{F}_q} \left( \frac{K_\mathbb{A}}{\Lambda(A) + K} \right),
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where \(A\) is an idele corresponding to \(A\) and \(\deg(A) \geq 2g - 1\). We see that

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for all ideles \(A\) with corresponding divisor \(A\) of degree at least \(2g - 1\).

With the above in mind, we certainly can say that for sufficiently large \(m \in \mathbb{Z}\), the divisor \(A = m \cdot v_0 - \sum_{i=1}^n (z_i + 1) \cdot v_i\) has a corresponding idele \(A\) such that \(K_\mathbb{A} = \Lambda(A) + K\).
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Notice that

\[ \text{ord}_{v_i}(\gamma) = \text{ord}_{v_i}(\gamma - \beta_i + \beta_i) = \text{ord}_{v_i}(\beta_i) = z_i \]

for all \( i = 1, \ldots, n \) by the strict triangle inequality and construction.
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Set $\alpha = \beta + \gamma$. Then exactly as above we see that

$$\text{ord}_{v_i}(\alpha - \alpha_i) = \text{ord}_{v_i}(\beta - \alpha_i + \gamma) = \text{ord}_{v_i}(\gamma) = z_i$$

for all $i = 1, \ldots, n$ by the strict triangle inequality and construction. We also have

$$\text{ord}_v(\alpha) = \text{ord}_v(\beta + \gamma) \geq \min\{\text{ord}_v(\beta), \text{ord}_v(\gamma)\} \geq 0$$

for all places $v \in S, v \neq v_1, \ldots, v_n$ by construction.