Hopefully you are familiar enough with vector spaces over fields to write down a correct definition on your own. Assuming that you write your scalar multiplication with the scalars on the left and the vectors on the right: $r \times v$, then the definition of a \textit{left} $R$-module for a (commutative with identity) ring $R$ is a snap. It’s the exact same definition, just with the scalars being elements of the ring $R$ instead of a field. With that in mind, perhaps the most obvious example of a left $R$-module is a vector space over a field, since certainly any field is a commutative ring with identity. Another obvious example is the ring $R$ itself, which may be viewed as a left (or right, for that matter) module over itself. Note that any abelian group $G$ may be viewed as a (left) $\mathbb{Z}$-module in the obvious manner.
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For example, consider $\mathbb{Z}$ as a module over itself. What does a submodule here look like? What does an ascending chain of submodules look like? For that matter, when considering a ring as a module over itself, what do the submodules look like? Here's an example of an infinite ascending chain of submodules.

Take the polynomial ring in (countably) infinitely many variables over a field $F$: $F[ X_1, X_2, ... ]$. Then $F[ X_1 ] \subset F[ X_1, X_2 ] \subset F[ X_1, X_2, X_3 ] \subset \cdots$
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Definition

A number field is a finite extension of the field of rational numbers \( \mathbb{Q} \).

Note that, as opposed to our (non)-example above, if \( \alpha \) is an algebraic integer in some number field, then \( \mathbb{Z}[\alpha] \) looks like it should be Noetherian. Indeed, we'll see that not only is this the case, but also the set of algebraic integers in a number field form a Noetherian ring (actually a Dedekind Domain!). This isn't as easy/obvious as it looks! For example, \( \sqrt{2} \) and \( \sqrt{3} \) are both clearly algebraic integers (with monic polynomials \( x^2 - 2 \) and \( x^2 - 3 \)). Is \( \sqrt{2} + \sqrt{3} \) an algebraic integer?
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Definition

A number field is a finite extension of the field of rational numbers \( \mathbb{Q} \). An element of such a field \( K \) is called an algebraic integer if it is a root of some monic polynomial in \( \mathbb{Z}[X] \).

Note that, as opposed to our (non)-example above, if \( \alpha \) is an algebraic integer in some number field, then \( \mathbb{Z}[\alpha] \) looks like it should be Noetherian. Indeed, we’ll see that not only is this the case, but also the set of algebraic integers in a number field form a Noetherian ring (actually a Dedekind Domain!).

This isn’t as easy/obvious as it looks! For example, \( \sqrt{2} \) and \( \sqrt{3} \) are both clearly algebraic integers (with monic polynomials \( X^2 - 2 \) and \( X^2 - 3 \)).

Is \( \sqrt{2} + \sqrt{3} \) an algebraic integer?