We will continue the notational conventions already established, with one slight wrinkle. We will denote ideles, i.e., elements of \( K \times A = \text{GL}_1(KA) \), with upper-case Latin letters: A, B, ... The associated divisors will be denoted using the corresponding fraktur font: A, B, ... In other words, if \( A = (A_v) \) is an idele, then the corresponding divisor is \( A = \text{div}((A_v)) = \sum_v v \in M(K) \text{ord}_v(A_v) \cdot v \).

Recall that for an idele \( A = (A_v) \), \( \Lambda(A) = \{ (b_v) \in KA : b_v \in A^{-1}O_v \text{ for all places } v \in M(K) \} \).

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The first step in our proof of the Riemann-Roch Theorem is the following proto-version.
Theorem (1)

Let $\mathbb{K}$ be a function field with field of constants $\mathbb{F}_{q}$. Then for all ideles $A \in GL_1(\mathbb{K}A)$ with corresponding divisor $A$, $l(A) = \deg(A) + 1 - g + \dim \mathbb{F}_{q}(\mathbb{K}A \Lambda(\mathbb{A}) + \mathbb{K}A)$. Where $g := \dim \mathbb{F}_{q}(\mathbb{K}A \Lambda(\mathbb{I}) + \mathbb{K}A)$.

Proof:

We claim that for ideles $A$ and $B$ satisfying $\Lambda(A) \supseteq \Lambda(B)$, $\dim \mathbb{F}_{q}(\Lambda(A)/\Lambda(B)) = \deg(A) - \deg(B)$. (1)
Theorem (1)

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$$l(\mathcal{A}) = \deg(\mathcal{A}) + 1 - g + \dim_{\mathbb{F}_q} \left( \frac{K_A}{\Lambda(A) + K} \right),$$

where $g$ is the genus of the function field $K$.
**Theorem (1)**

Let $K$ be a function field with field of constants $\mathbb{F}_q$. Then for all ideles $A \in \text{GL}_1(K_A)$ with corresponding divisor $\mathfrak{A}$,

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Theorem (1)

Let $K$ be a function field with field of constants $\overline{\mathbb{F}}_q$. Then for all ideles $A \in \text{GL}_1(K_{\mathfrak{A}})$ with corresponding divisor $\mathfrak{A}$,

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Proof:
Theorem (1)

Let \( K \) be a function field with field of constants \( \mathbb{F}_q \). Then for all ideles \( A \in \text{GL}_1(K_A) \) with corresponding divisor \( \mathfrak{A} \),

\[
l(\mathfrak{A}) = \deg(\mathfrak{A}) + 1 - g + \dim_{\mathbb{F}_q} \left( \frac{K_A}{\Lambda(A) + K} \right),
\]

where

\[
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**Proof:** We claim that for ideles \( A \) and \( B \) satisfying \( \Lambda(A) \supseteq \Lambda(B) \),
Theorem (1)

Let $K$ be a function field with field of constants $\mathbb{F}_q$. Then for all ideles $A \in \text{GL}_1(K_A)$ with corresponding divisor $\mathcal{A}$,

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To see why, it suffices to consider the case where $A_v^{-1} \mathfrak{O}_v = B_v^{-1} \mathfrak{O}_v$ for all but one place.
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Combining these observation with (1), we get

\[ \text{deg}(A) - \text{deg}(B) = l(A) - l(B) + \dim F_q \left( \Lambda(A) + K \Lambda(B) + K \right). \]
Combining these observation with (1), we get

\[ \deg(A) - \deg(B) = l(A) - l(B) + \dim_{\mathbb{F}_q} \left( \frac{\Lambda(A) + K}{\Lambda(B) + K} \right). \]  

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Combining these observations with (1), we get

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Combining these observation with (1), we get

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Let \( B \) be an arbitrary idele and choose an idele \( A \) with \( \Lambda(A) = \Lambda(B) + \Lambda(C) \).
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Combining these observation with (1), we get

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In particular, the last summand above is finite,
Combining these observation with (1), we get

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\[ \deg(A) - \deg(B) = l(A) - l(B) + \text{dim}_{\mathbb{F}_q} \left( \frac{K_A}{\Lambda(B) + K} \right). \]

(3)

In particular, the last summand above is finite, so that we may rewrite (2) in the form

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Combining these observation with (1), we get

\[ \deg(\mathcal{A}) - \deg(\mathcal{B}) = l(\mathcal{A}) - l(\mathcal{B}) + \dim_{\mathbb{F}_q} \left( \frac{\Lambda(A) + K}{\Lambda(B) + K} \right). \] \hspace{1cm} (2)

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In particular, the last summand above is finite, so that we may rewrite (2) in the form

\[ \deg(\mathcal{A}) - \deg(\mathcal{B}) = l(\mathcal{A}) - l(\mathcal{B}) + \dim_{\mathbb{F}_q} \left( \frac{K_A}{\Lambda(B) + K} \right) - \dim_{\mathbb{F}_q} \left( \frac{K_A}{\Lambda(A) + K} \right). \] \hspace{1cm} (3)
Now let $A$ and $B$ be arbitrary ideles and choose an idele $D$ satisfying both $\Lambda(D) \supseteq \Lambda(A)$ and $\Lambda(D) \supseteq \Lambda(B)$.
Now let $A$ and $B$ be arbitrary ideles and choose an idele $D$ satisfying both $\Lambda(D) \supseteq \Lambda(A)$ and $\Lambda(D) \supseteq \Lambda(B)$. Then we may use (3) on both pairs of ideles: 

\[-\deg(A) - \dim \mathbb{F}_q(K_A \Lambda(A) + K_A) + \lambda(A) = -\deg(D) - \dim \mathbb{F}_q(K_D \Lambda(D) + K_A) + \lambda(D) = -\deg(B) - \dim \mathbb{F}_q(K_D \Lambda(B) + K_A) + \lambda(B).\]
Now let $A$ and $B$ be arbitrary ideles and choose an idele $D$ satisfying both $\Lambda(D) \supseteq \Lambda(A)$ and $\Lambda(D) \supseteq \Lambda(B)$. Then we may use (3) on both pairs of ideles: $D$ and $A$, 

\[ -\deg(A) - \dim F_q(K_A \Lambda(A) + K_A) + l(A) = -\deg(D) - \dim F_q(K_A \Lambda(D) + K_A) + l(D) \]

This shows that for any idele $A$, the quantity 

\[ -\deg(A) - \dim F_q(K_A \Lambda(A) + K_A) + l(A) \]

is the same.
Now let $A$ and $B$ be arbitrary ideles and choose an idele $D$ satisfying both $\Lambda(D) \supseteq \Lambda(A)$ and $\Lambda(D) \supseteq \Lambda(B)$. Then we may use (3) on both pairs of ideles: $D$ and $A$, and $D$ and $B$. 

\[ -\deg(A) - \dim_{F_q} \left( K_A \Lambda(A) + K_A \right) + l(A) = -\deg(D) - \dim_{F_q} \left( K_A \Lambda(D) + K_A \right) + l(D) \]

This shows that for any idele $A$, the quantity $-\deg(A) - \dim_{F_q} \left( K_A \Lambda(A) + K_A \right) + l(A)$ is the same.
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Using (3) twice now yields the following:

$$-\deg(\mathcal{A}) - \dim_{\mathbb{F}_q} \left( \frac{K_A}{\Lambda(A) + K} \right) + l(\mathcal{A})$$
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\[
- \deg(\mathcal{U}) - \dim_{F_q} \left( \frac{K_A}{\Lambda(A) + K} \right) + l(\mathcal{U})
= - \deg(\mathcal{D}) - \dim_{F_q} \left( \frac{K_A}{\Lambda(D) + K} \right) + l(\mathcal{D})
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Now let $A$ and $B$ be arbitrary ideles and choose an idele $D$ satisfying both $\Lambda(D) \supseteq \Lambda(A)$ and $\Lambda(D) \supseteq \Lambda(B)$. Then we may use (3) on both pairs of ideles: $D$ and $A$, and $D$ and $B$. Using (3) twice now yields the following:

$$- \deg(A) - \dim_{\mathbb{F}_q} \left( \frac{K_A}{\Lambda(A) + K} \right) + l(A)$$

$$= - \deg(D) - \dim_{\mathbb{F}_q} \left( \frac{K_A}{\Lambda(D) + K} \right) + l(D)$$

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Using (3) twice now yields the following:

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$$= - \deg(\mathcal{D}) - \dim_{\mathbb{F}_q} \left( \frac{K_A}{\Lambda(D) + K} \right) + l(\mathcal{D})$$

$$= - \deg(\mathcal{B}) - \dim_{\mathbb{F}_q} \left( \frac{K_A}{\Lambda(B) + K} \right) + l(\mathcal{B}).$$

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Now let $A$ and $B$ be arbitrary ideles and choose an idele $D$ satisfying both $\Lambda(D) \supseteq \Lambda(A)$ and $\Lambda(D) \supseteq \Lambda(B)$. Then we may use (3) on both pairs of ideles: $D$ and $A$, and $D$ and $B$.

Using (3) twice now yields the following:

\[- \deg(\mathcal{A}) - \dim_{F_q} \left( \frac{K_A}{\Lambda(A) + K} \right) + I(\mathcal{A}) = - \deg(\mathcal{D}) - \dim_{F_q} \left( \frac{K_A}{\Lambda(D) + K} \right) + I(\mathcal{D}) = - \deg(\mathcal{B}) - \dim_{F_q} \left( \frac{K_A}{\Lambda(B) + K} \right) + I(\mathcal{B}).\]

This shows that for any idele $A$, the quantity

\[- \deg(\mathcal{A}) - \dim_{F_q} \left( \frac{K_A}{\Lambda(A) + K} \right) + I(\mathcal{A})\]

is the same.
As a particular case, we can consider the identity idele $I$. 

\[ \text{The corresponding divisor is obviously 0. We then get (recall that we previously proved $l(0) = 1$)} \]

\[ -\deg(A) - \dim_{Fq}(K_A + K_A) + l(A) = -\deg(0) - \dim_{Fq}(K_A + K_A) + l(0) = 0 - g + 1. \]

Rearranging yields Theorem 1.
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\[
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\
= \
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As a particular case, we can consider the identity idele \( I \). The corresponding divisor is obviously 0.

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\[
- \deg(\mathcal{A}) - \dim_{\mathbb{F}_q} \left( \frac{K_A}{\Lambda(A) + K} \right) + l(\mathcal{A})
\]

\[
= - \deg(0) - \dim_{\mathbb{F}_q} \left( \frac{K_A}{\Lambda(I) + K} \right) + l(0)
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As a particular case, we can consider the identity idele $I$. The corresponding divisor is obviously 0.

We then get (recall that we previously proved $I(0) = 1$)

$$- \deg(\mathcal{A}) - \dim_{\mathbb{F}_q} \left( \frac{K_{\mathcal{A}}}{\Lambda(A) + K} \right) + I(\mathcal{A})$$

$$= - \deg(0) - \dim_{\mathbb{F}_q} \left( \frac{K_{\mathcal{A}}}{\Lambda(I) + K} \right) + I(0)$$

$$= 0 - g + 1.$$
As a particular case, we can consider the identity idele $I$. The corresponding divisor is obviously 0.

We then get (recall that we previously proved $l(0) = 1$)

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Rearranging yields Theorem 1.
Theorem (Riemann’s Theorem)

The genus \( g \) may be characterized as the maximum of \( \deg(A) - l(A) + 1 \) over all divisors \( A \in \text{Div}(K) \).

Further, there is an integer \( z \), depending only on \( K \), such that \( \deg(A) - l(A) + 1 = g \) for all divisors \( A \) with \( \deg(A) \geq z \).

Proof: By Theorem 1 we have \( \deg(A) - l(A) + 1 = g - \dim \mathbb{F}^q(K_A \Lambda(A) + K) \) for all divisors \( A \).

The first part of Riemann's Theorem is thus that if a divisor \( A \) with \( K_A = \Lambda(A) + K \), which we proved on Monday. Now choose a divisor \( A_0 \) with \( K_{A_0} = \Lambda(A_0) + K \), i.e., one which satisfies \( \deg(A_0) - l(A_0) + 1 = g \), and set \( z = \deg(A_0) + g \). Then for any divisor \( A \) with \( \deg(A) \geq z \) we have \( l(A - A_0) \geq \deg(A - A_0) + 1 - g \geq z - \deg(A_0) + 1 - g \geq 1 \).
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Proof:

By Theorem 1 we have $\deg(A) - l(A) + 1 = g - \dim \mathbb{F}_q(\Lambda(A) + K)$ for all divisors $A$. The first part of Riemann’s Theorem is thus that there exists a divisor $A$ with $K_A = \Lambda(A) + K$, which we proved on Monday. Now choose a divisor $A_0$ with $K_{A_0} = \Lambda(A_0) + K$, i.e., one which satisfies $\deg(A_0) - l(A_0) + 1 = g$, and set $z = \deg(A_0) + g$. Then for any divisor $A$ with $\deg(A) \geq z$ we have $l(A - A_0) \geq \deg(A - A_0) + 1 - g \geq z - \deg(A_0) + 1 - g \geq 1$. 

Math 681, Monday, March 15

March 17, 2021
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Theorem (Riemann’s Theorem)

The genus $g$ may be characterized as the maximum of $\deg(\mathfrak{A}) - l(\mathfrak{A}) + 1$ over all divisors $\mathfrak{A} \in \text{Div}(K)$. Further, there is an integer $z$, depending only on $K$, such that $\deg(\mathfrak{A}) - l(\mathfrak{A}) + 1 = g$ for all divisors $\mathfrak{A}$ with $\deg(\mathfrak{A}) \geq z$.

Proof: By Theorem 1 we have

$$\deg(\mathfrak{A}) - l(\mathfrak{A}) + 1 = g - \dim_{\mathbb{F}_q} \left( \frac{K_\mathfrak{A}}{\Lambda(A) + K} \right)$$

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**Theorem (Riemann’s Theorem)**

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$$l(A - A_0) \geq \deg(A - A_0) + 1 - g \geq z - \deg(A_0) + 1 - g \geq 1.$$
The above shows that $L(\mathcal{A} - \mathcal{A}_0) \neq \{0\},$
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We now have by (3) and the containment $\Lambda(\mathcal{B}) + K \supseteq \Lambda(\mathcal{A}_0) + K$
The above shows that $L(A - A_0) \neq \{0\}$, so take a non-zero $\alpha \in L(A - A_0)$ and consider $B = A + \text{div}(\alpha)$; this divisor is linearly equivalent to $A$ and satisfies $B \geq A_0$.

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\[
\deg(\mathcal{A}) - \ell(\mathcal{A}) = \deg(\mathcal{B}) - \ell(\mathcal{B}) = \deg(\mathcal{A}_0) - \ell(\mathcal{A}_0) + \dim_{\mathbb{F}_q} \left( \frac{K_{\mathcal{A}}}{\Lambda(\mathcal{A}_0) + K} \right) - \dim_{\mathbb{F}_q} \left( \frac{K_{\mathcal{A}}}{\Lambda(\mathcal{B}) + K} \right)
\]

This completes our proof of Riemann's Theorem.
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$$\text{deg}(A) - l(A) = \text{deg}(B) - l(B)$$
$$= \text{deg}(A_0) - l(A_0)$$
$$+ \dim_{\mathbb{F}_q} \left( \frac{K_A}{\Lambda(A_0) + K} \right) - \dim_{\mathbb{F}_q} \left( \frac{K_A}{\Lambda(B) + K} \right)$$
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\deg(\mathcal{A}) - l(\mathcal{A}) = \deg(\mathcal{B}) - l(\mathcal{B})
\]
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\]
\[
+ \dim_{F_q} \left( \frac{K_{\mathcal{A}}}{\Lambda(\mathcal{A}_0) + K} \right) - \dim_{F_q} \left( \frac{K_{\mathcal{A}}}{\Lambda(\mathcal{B}) + K} \right)
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\geq \deg(\mathcal{A}_0) - l(\mathcal{A}_0)
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\[
\deg(\mathcal{A}) - l(\mathcal{A}) = \deg(\mathcal{B}) - l(\mathcal{B}) \\
= \deg(\mathcal{A}_0) - l(\mathcal{A}_0) \\
+ \dim_{\mathbb{F}_q} \left( \frac{K_{\mathcal{A}}}{\Lambda(A_0) + K} \right) - \dim_{\mathbb{F}_q} \left( \frac{K_{\mathcal{A}}}{\Lambda(B) + K} \right) \\
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This completes our proof of Riemann’s Theorem.
Recall the notion of the (algebraic) *dual space* of a vector space:
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The Riemann-Roch Theorem is a consequence of Theorem 1 together with an appropriate realization of $K_A/\left(\Lambda(A) + K\right)$ as a dual space.
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For a finite-dimensional vector space
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We’ll take the most direct approach possible
Recall the notion of the (algebraic) dual space of a vector space: if $V$ is a vector space over a field $F$, then the dual space $V'$ consists of the linear transformations from $V$ into $F$.

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The Riemann-Roch Theorem is a consequence of Theorem 1 together with an appropriate realization of $K_A/(\Lambda(A) + K)$ as a dual space.

We’ll take the most direct approach possible (given what we’ve already proven),
Recall the notion of the (algebraic) *dual space* of a vector space: if $V$ is a vector space over a field $F$, then the dual space $V'$ consists of the linear transformations from $V$ into $F$.

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The Riemann-Roch Theorem is a consequence of Theorem 1 together with an appropriate realization of $K_A/\left(\Lambda(A) + K\right)$ as a dual space.

We’ll take the most direct approach possible (given what we’ve already proven), but do note that there are many approaches,
Recall the notion of the (algebraic) dual space of a vector space: if $V$ is a vector space over a field $F$, then the dual space $V'$ consists of the linear transformations from $V$ into $F$.

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We’ll take the most direct approach possible (given what we’ve already proven), but do note that there are many approaches, each with their own attributes.
Definition

A Weil differential is an $F_q$-linear transformation $\omega: K_A \to F_q$ whose kernel contains a subset of the form $\Lambda(A) + K$ for some idele $A$.

For a given idele $A$ we denote the collection of Weil differentials vanishing on $\Lambda(A) + K$ by $\Omega_K(A)$, and denote the set of all Weil differentials by $\Omega_K$.

It's a trivial matter to confirm that $\Omega_K$ is a subspace of the dual space of $K_A$ (viewed as a vector space over $F_q$ in the obvious manner).

In fact, it's clear that $\Omega_K(A)$ is the dual space of $K_A / (\Lambda(A) + K)$, so that $\dim F_q(\Omega_K(A)) = \dim F_q(K_A / (\Lambda(A) + K))$.

Directly from the definition we see that $\Omega_K(A) \subseteq \Omega_K(B)$ whenever the associated divisors satisfy $A \leq B$, implying $\Lambda(A) \subseteq \Lambda(B)$.
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(4)

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A **Weil differential** is an \( \mathbb{F}_q \)-linear transformation \( \omega : K_A \to \mathbb{F}_q \) whose kernel contains a subset of the form \( \Lambda(A) + K \) for some idele \( A \). For a given idele \( A \) we denote the collection of Weil differentials vanishing on \( \Lambda(A) + K \) by \( \Omega_K(A) \), and denote the set of all Weil differentials by \( \Omega_K \).

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Directly from the definition we see that \( \Omega_K(A) \subseteq \Omega_K(B) \) whenever the associated divisors satisfy \( A \leq B \), implying \( \Lambda(A) \subseteq \Lambda(B) \).
We may also view $\Omega_K$ as a vector space over $K$ as follows.

For $\alpha \in K$ and $\omega \in \Omega_K$, set

$$\alpha \omega \left( (a_v) \right) = \omega \left( \alpha a_v \right)$$

for all $(a_v) \in K A$.

In particular, note that if $\alpha \neq 0$ and $\omega$ vanishes on $\Lambda(A) + K$, then $\alpha \omega$ vanishes on $\Lambda(\alpha A) + K$.

**Proposition (1)**

As a vector space over $K$, $\Omega_K$ has dimension 1.

**Proof:**

We first note that $\Omega_K \neq \{0\}$. Indeed, by Theorem 1 we must have a non-zero element of $\Omega_K(A)$ whenever the associated divisor $A$ has degree less than $-1$, say.
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For $\alpha \in K$ and $\omega \in \Omega_K$,

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As for the claim, let \( z \) be the quantity in Riemann’s Theorem and let \( B \) be such that

\[
\deg(B) \geq \max\{z - \deg(A), \ z - \deg(A'), \ 1, \ 3(g - 1) - \deg(A) - \deg(A')\}.
\]

Then both \( \deg(B + A) \), \( \deg(B + A') \) \( \geq z \), so by Riemann’s Theorem

\[
\begin{align*}
\l(A + B) &= \deg(A) + 1 - g + \deg(B), \\
\l(A' + B) &= \deg(A') + 1 - g + \deg(B).
\end{align*}
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On the other hand, by Theorem 1 and (4),
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Then both $\deg(B + A), \ \deg(B + A') \geq z$, so by Riemann’s Theorem

$$l(A + B) = \deg(A) + 1 - g + \deg(B),$$

$$l(A' + B) = \deg(A') + 1 - g + \deg(B). \quad (5)$$

On the other hand, by Theorem 1 and (4), $\Omega_K(B^{-1})$ has dimension
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On the other hand, by Theorem 1 and (4), \( \Omega_K(B^{-1}) \) has dimension

\[
\dim_{\mathbb{F}_q} (\Omega_K(B^{-1})) = l(-B) - \deg(-B) - 1 + g = \deg(B) - 1 + g
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We claim that one can choose $B$ such that the images of $\phi$ and $\phi'$ have non-trivial intersection. Note that, given this claim, we then have $\alpha, \alpha' \in K^\times$ with $\alpha \omega = \alpha' \omega'$, so that $\omega' \in \omega K$. In other words, the proof of Proposition 1 follows from this claim.

As for the claim, let $z$ be the quantity in Riemann's Theorem and let $B$ be such that

$$\deg(B) \geq \max\{z - \deg(A), z - \deg(A'), 1, 3(g - 1) - \deg(A) - \deg(A')\}.$$  

Then both $\deg(B + A), \deg(B + A') \geq z$, so by Riemann's Theorem

$$l(A + B) = \deg(A) + 1 - g + \deg(B),$$
$$l(A' + B) = \deg(A') + 1 - g + \deg(B).$$

On the other hand, by Theorem 1 and (4), $\Omega_K(B^{-1})$ has dimension

$$\dim_{\mathbb{F}_q}(\Omega_K(B^{-1})) = l(-B) - \deg(-B) - 1 + g = \deg(B) - 1 + g$$

since $\deg(-B) = -\deg(B) \leq -1$. 

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Math 681, Monday, March 15

March 17, 2021
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This proves that these images have non-trivial intersection.
Now (5) tells us the dimensions (as $\mathbb{F}_q$-vector spaces) of the respective images of $\phi$ and $\phi'$, which when added are larger than the dimension of the codomain $\Omega_K(B^{-1})$ by (6) and construction.

This proves that these images have non-trivial intersection. That proves our claim, and whence Proposition 1.