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Via the diagonal embedding as before, the field $K$ is a discrete subset of the adele ring $K_A$ and the quotient $K_A/K$ is compact.

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We’ll start today with the case $K = \mathbb{F}_p(\mathbb{X})$. 

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The Adele Ring (cont.)

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Similar to how we proceeded in the case $K = \mathbb{Q}$, for every place $v \in M(K)$ set $K^{(v)}$ to be the subset of $\alpha \in K$ where $|\alpha|_w \leq 1$ for all places $w \neq v$. We then have

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Any $\alpha \in K^{(v)} \cap \mathcal{O}_v$ satisfies $|\alpha|_w \leq 1$ for all places $w \in M(K)$. 
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Any $\alpha \in K^{(v)} \cap \mathcal{O}_v$ satisfies $|\alpha|_w \leq 1$ for all places $w \in M(K)$. We’ve already seen that the only such $\alpha$ are elements of $\mathbb{F}_p$ here.
Using (1),

$$K_A = K_A(\emptyset) + K_A,$$

$$K_A(\emptyset) \cap K_A = F_p,$$

where we have identified $$F_p \subset K \subset K_A$$ via the diagonal embedding above.

To see why, let $$(\alpha_v)_v \in K_A$$ where $${\alpha_v} \in K_v$$ for all $$v \in M(K)$$ and $${\alpha_v} \in O_v$$ for all but finitely many $$v$$.

Just as in the case $$K = Q$$ before, (2) suffices to prove the Theorem in the case $$K = F_p(X)$$.
Using (1), we claim that

\[ K_\mathbb{A} = K_\mathbb{A}(\emptyset) + K, \quad K_\mathbb{A}(\emptyset) \cap K = \mathbb{F}_p, \]  

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Using (1), we claim that

$$K_\Delta = K_\Delta(\emptyset) + K, \quad K_\Delta(\emptyset) \cap K = \mathbb{F}_p,$$

where we have identified $\mathbb{F}_p \subset K \subset K_\Delta$ via the diagonal embedding above.

To see why, let $(\alpha_v)_v \in K_\Delta$ where $\alpha_v \in K_v$ for all $v \in M(K)$ and $\alpha_v \in \mathcal{O}_v$ for all but finitely many $v$.

Just as in the case $K = \mathbb{Q}$ before,
Using (1), we claim that

\[ K_{\mathbb{A}} = K_{\mathbb{A}}(\emptyset) + K, \quad K_{\mathbb{A}}(\emptyset) \cap K = \mathbb{F}_p, \tag{2} \]

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To see why, let \( (\alpha_v)_v \in K_{\mathbb{A}} \) where \( \alpha_v \in K_v \) for all \( v \in M(K) \) and \( \alpha_v \in \mathcal{O}_v \) for all but finitely many \( v \).

Just as in the case \( K = \mathbb{Q} \) before, (2) suffices to prove the Theorem in the case \( K = \mathbb{F}_p(X) \).
The idele group is the group \( K \times A \), the (multiplicatively) invertible elements of the adele ring.

**Lemma (5)**
The idele group consists of those adeles \((\alpha_v)_v \in M(K)\) where \(\alpha_v \neq 0\) for all places and \(|\alpha_v|_v = 1\) for all but finitely many places.

In particular, \(K^\times\) (the non-zero elements of the field \(K\)) is a subgroup of the idele group via the diagonal embedding.

**Proof:**
Let \((\alpha_v)_v \in K \times A\). Then by definition there is a \((\beta_v)_v \in K^A\) such that \((\alpha_v) \cdot (\beta_v) = 1\).

In particular, \(\alpha_v \beta_v = 1\) for all places \(v\), so that \(\alpha_v \neq 0\).

Further, \(|\beta_v|_v \leq 1\) for all but finitely many places \(v\) and the same goes for \(\alpha_v\).

Therefore \(|\alpha_v|_v = 1 = |\beta_v|_v\) for all but finitely many places \(v \in M(K)\).
Definition

The *idele group* is the group $K_A^\times$, where $K_A^\times$ consists of those adeles $(\alpha_v)_{v \in \mathcal{M}(K)}$ such that $\alpha_v \neq 0$ for all places and $|\alpha_v|_v = 1$ for all but finitely many places. In particular, $K^\times$ (the non-zero elements of the field $K$) is a subgroup of the idele group via the diagonal embedding.

Proof:

Let $(\alpha_v) \in K_A^\times$. Then by definition there is a $(\beta_v) \in K_A$ such that $(\alpha_v) \cdot (\beta_v) = 1$. In particular, $\alpha_v \beta_v = 1$ for all places $v$, so that $\alpha_v \neq 0$. Further, $|\beta_v|_v \leq 1$ for all but finitely many places $v$ and the same goes for $\alpha_v$. Therefore $|\alpha_v|_v = 1 = |\beta_v|_v$ for all but finitely many places $v \in \mathcal{M}(K)$.
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Let $(\alpha_v) \in \mathbb{K}_A^\times$. Then by definition there is a $(\beta_v) \in \mathbb{K}_A$ such that $(\alpha_v) \cdot (\beta_v) = 1$. In particular, $\alpha_v \beta_v = 1$ for all places $v$, so that $\alpha_v \neq 0$. Further, $|\beta_v|_v \leq 1$ for all but finitely many places $v$ and the same goes for $\alpha_v$. Therefore $|\alpha_v|_v = 1 = |\beta_v|_v$ for all but finitely many places $v \in M(\mathbb{K})$. 
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**Proof:** Let $(\alpha_v) \in K_A^\times$. Then by definition there is a $(\beta_v) \in K_A$ such that $(\alpha_v) \cdot (\beta_v) = 1$. 
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Proof: Let $(\alpha_v) \in K^\times_\mathbb{A}$. Then by definition there is a $(\beta_v) \in K_\mathbb{A}$ such that $(\alpha_v) \cdot (\beta_v) = 1$. In particular, $\alpha_v \beta_v = 1$ for all places $v$, 

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**Proof:** Let $(\alpha_v) \in K^{\times}_{\mathbb{A}}$. Then by definition there is a $(\beta_v) \in K_{\mathbb{A}}$ such that $(\alpha_v) \cdot (\beta_v) = 1$. In particular, $\alpha_v \beta_v = 1$ for all places $v$, so that $\alpha_v \neq 0$.  

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**Proof**: Let \( (\alpha_v) \in \mathbb{A}_K^\times \). Then by definition there is a \( (\beta_v) \in \mathbb{A}_K \) such that \( (\alpha_v) \cdot (\beta_v) = 1 \). In particular, \( \alpha_v \beta_v = 1 \) for all places \( v \), so that \( \alpha_v \neq 0 \). Further, \( |\beta_v|_v \leq 1 \) for all but finitely many places \( v \).
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Recall the definitions of ramification index and residue class degree for number fields. In the language of places/valuation rings, the residue class degree of a non-archimedean place \( v \in M(\mathbb{K}) \) is the degree of the extension \( O_P/M_P \) over \( O_p/M_p \equiv \mathbb{Z}/p\mathbb{Z} \), where \( p \in M(\mathbb{Q}) \) is the prime/place lying below \( v \).

The ramification index may be viewed as the positive integer \( e \) where \( \text{ord}_v(a) = e \text{ord}_p(a) \) for all non-zero \( a \in \mathbb{Q} \subseteq \mathbb{K} \).

We thus see that both the residue class degree and ramification index of a place of a function field can be given in the exact same well-defined manner; just use \( \mathbb{F}_p(X) \) in place of \( \mathbb{Q} \).

It's handy to define the ramification index of any archimedean place to be 1 and the residue class degree to be 1 if it corresponds to a real embedding, and 2 if it corresponds to a pair of complex conjugate embeddings.
Recall the definitions of *ramification index* and *residue class degree* for number fields.

In the language of places/valuation rings, the residue class degree of a non-archimedean place \( v \in M(K) \) is the degree of the extension \( \mathcal{O}_P/M_P \cong \mathbb{Z}/p\mathbb{Z} \), where \( p \in M(\mathbb{Q}) \) is the prime/place lying below \( v \).

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The ramification index may be viewed as the positive integer $e$ where $\text{ord}_v(a) = e \text{ord}_p(a)$ for all non-zero $a \in \mathbb{Q} \subseteq K$.

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Math 681, Wednesday, March 3 and Friday, March 5
The Artin-Whaples Product Formula

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Theorem
For all places $w \in M(\mathbb{Q})$ and all number fields $K$ we have

$$\sum_{v \in M(K)} v|w = [K:\mathbb{Q}].$$

Definition
For a number field $K$ and a place $v \in M(K)$, the local degree $n_v = e_v f_v$, the product of the ramification index and residue class degree.

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Theorem (Product Formula)

Let $K$ be a number field. For all $v \in \mathcal{M}(K)$ let $|\cdot|_v$ be the unique absolute value on $K$ that extends $|\cdot|_w$ on $\mathbb{Q}$, where $w$ is the place of $\mathbb{Q}$ lying below $v$ (i.e., $v|w$). Then for all non-zero $\alpha \in K$ we have

$$\prod_{v \in \mathcal{M}(K)} |\alpha|_v = 1.$$

Proof: Since $K$ is the quotient field of $\mathcal{O}_K$, it suffices to prove this for algebraic integers. Let $\alpha$ be a non-zero element of $\mathcal{O}_K$. For a place $v|\infty$ we have $|\alpha|_v = |\sigma(\alpha)|_\infty$, where $\sigma$ is the associated embedding of $K$ into $\mathbb{C}$ and $|\cdot|_\infty$ is the usual complex modulus. Then by the definitions

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For a place $v|\infty$ we have $|\alpha|_v = |\sigma(\alpha)|_\infty$, where $\sigma$ varies over embeddings of $K$ into $C$. For a place $v|\mathbb{Q}$, we can use the Galois theory of $K/\mathbb{Q}$ to show that $|\alpha|_v = |\sigma(\alpha)|_v$ for all embeddings $\sigma$ of $K$ into $C$.

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For a non-archimedean place $v$ lying above a prime $p$. 

Further, $\text{ord}_v(p) = e_v$. 

This implies that $|\alpha|_v = \text{N}(P) - \text{ord}_v(\alpha)/n_v$ for all $\alpha \in K^\times$. 

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We further note that the norm is a group homomorphism from the (multiplicative) group of non-zero fractional ideals into $\mathbb{Q}^\times$. 
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This leads us to the following.
Definition

For a function field $K$, the divisor group $\text{Div}(K)$ is the free abelian group generated by the places $M(K)$. Elements of this group, called divisors, are written additively:

$$A = \sum_{v \in M(K)} z_v \cdot v,$$

where $z_v \in \mathbb{Z}$, and $z_v = 0$ almost always (here "almost always" means "for all but finitely many"). The integer coefficients $z_v$ are the order of the divisor at the place $v$ and written $\text{ord}_v(A) = z_v$.

Note how any idele $(\alpha_v)_v \in K \times A$ leads to a divisor:

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where \( z_v \in \mathbb{Z} \), and \( z_v = 0 \) almost always (here “almost always” means “for all but finitely many”).

The integer coefficients \( z_v \) are the order of the divisor at the place \( v \) and written \( \text{ord}_v(A) = z_v \).

Note how any idele \((\alpha_v)_v \in K^\times A\) leads to a divisor:

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\text{div}\left((\alpha_v)_v\right) = \sum_{v \in M(K)} \text{ord}_v(\alpha_v) v.
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Recall the definition of the degree of a place/valuation ring for a function field:

\[ \deg(v) = \left[ \frac{\mathbb{R}}{M_F} \right]. \]

(Here \( p \) is the characteristic of \( K \); \( \deg(v) \) is some multiple of the residue class degree.)

This degree function can be extended to \( \text{Div}(K) \) giving a group homomorphism \( \deg : \text{Div}(K) \to \mathbb{Z} \) via

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Whither the analogs of principal (fractional) ideals?
Proposition

Let $K$ be a function field. For every non-zero $\alpha \in K$ we get a principal divisor $\text{div}(\alpha) := \sum_{v \in M(K)} \text{ord}_v(\alpha) \cdot v$. That is to say, there are only finitely many places $v \in M(K)$ for which $\text{ord}_v(\alpha) \neq 0$. Indeed, for such an $\alpha$ the zero divisor and pole divisor $\text{div}(\alpha) + = \sum_{v \in M(K)} \text{ord}_v(\alpha) > 0 \text{ord}_v(\alpha) \cdot v$, $\text{div}(\alpha) - = \sum_{v \in M(K)} \text{ord}_v(\alpha) < 0 \text{ord}_v(\alpha) \cdot v$, respectively, satisfy $\deg(\text{div}(\alpha) +), \deg(\text{div}(\alpha) -) \leq [K: F_p(\alpha)]$. 

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respectively, satisfy

$$\deg (\text{div}(\alpha)^+), \quad \deg (\text{div}(\alpha)^-) \leq [K : \mathbb{F}_p(\alpha)].$$
Proof:

We will show that if \( v_1, \ldots, v_r \in M(K) \) with \( \text{ord}_{v_i}(\alpha) > 0 \) for all, then

\[
\sum_{i=1}^{r} \text{ord}_{v_i}(\alpha) \deg(v_i) \leq [K:F]_{\mathbb{P}(\alpha)}.
\]

(5)

For notational convenience set \( e_i = \text{ord}_{v_i}(\alpha) \) and \( f_i = \deg(v_i) \).

For each \( i = 1, \ldots, r \), we use the Weak Approximation Theorem to prove the existence of \( \pi_i \in K \) with

\[
\text{ord}_{v_j}(\pi_i) = \begin{cases} 
1 & \text{if } i = j, \\
0 & \text{otherwise}.
\end{cases}
\]

Next, for each \( i = 1, \ldots, r \), choose \( \beta_{i,1}, \ldots, \beta_{i,f_i} \in R_i \) with \( R_i/M_i = \{ \beta_{i,j} + M_i : j = 1, \ldots, f_i \} \), where \( R_i \) is the valuation ring for \( v_i \) with maximal ideal \( M_i \).
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For notational convenience set \( e_i = \text{ord}_{v_i}(\alpha) \) and \( f_i = \deg(v_i) \). For each \( i = 1, \ldots, r \) we use the Weak Approximation Theorem to prove the existence of \( \pi_i \in K \) with
\[
\text{ord}_{v_j}(\pi_i) = \begin{cases} 
1 & \text{if } i = j, \\
0 & \text{otherwise.}
\end{cases}
\]

Next, for each \( i = 1, \ldots, r \) choose \( \beta_{i,1}, \ldots, \beta_{i,f_i} \in R_i \) with
Proof: We will show that if $\nu_1, \ldots, \nu_r \in M(K)$ with $\text{ord}_{\nu_i}(\alpha) > 0$ for all, then

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For notational convenience set $e_i = \text{ord}_{\nu_i}(\alpha)$ and $f_i = \deg(\nu_i)$. For each $i = 1, \ldots, r$ we use the Weak Approximation Theorem to prove the existence of $\pi_i \in K$ with

$$\text{ord}_{\nu_j}(\pi_i) = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{otherwise.} \end{cases}$$

Next, for each $i = 1, \ldots, r$ choose $\beta_{i,1}, \ldots, \beta_{i,f_i} \in R_i$ with

$$R_i/M_i = \{\beta_{i,j} + M_i : j = 1, \ldots, f_i\},$$
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\[
R_i/\mathcal{M}_i = \{\beta_{i,j} + \mathcal{M}_i : j = 1, \ldots, f_i\},
\]
where $R_i$ is the valuation ring for $\nu_i$ with maximal ideal $\mathcal{M}_i$. 

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By another application of the Weak Approximation Theorem
By another application of the Weak Approximation Theorem there are \( \alpha_{i,j} \in K, \ 1 \leq i \leq r, \ 1 \leq j \leq f_i \) satisfying

\[
\text{ord}_v \left( \beta_{i,j} - \alpha_{i,j} \right) > 0 \quad \text{and} \quad \text{ord}_v \left( \alpha_{i,j} \right) \geq e_i \quad \text{for all} \ l \neq i.
\]

We claim that \( \{ \pi_n^i \alpha_{i,j} : 1 \leq i \leq r, \ 1 \leq j \leq f_i, \ 0 \leq n \leq e_i \} \) is linearly independent over \( F_p(\alpha) \).

Note that (5) follows directly from this claim.
By another application of the Weak Approximation Theorem there are \( \alpha_{i,j} \in K \), \( 1 \leq i \leq r \), \( 1 \leq j \leq f_i \) satisfying \( \text{ord}_{v_i}(\beta_{i,j} - \alpha_{i,j}) > 0 \)
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\( \text{ord}_{v_i}(\beta_{i,j} - \alpha_{i,j}) > 0 \) and 
\( \text{ord}_{v_l}(\alpha_{i,j}) \geq e_l \) for all \( l \neq i \).
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$$\left\{ \pi_i^n \alpha_{i,j} : 1 \leq i \leq r, \ 1 \leq j \leq f_i, \ 0 \leq n \leq e_i \right\}$$

is linearly independent over $\mathbb{F}_p(\alpha)$. Note that (5) follows directly from this claim.
Suppose we have a non-trivial linear combination

\[
\sum_{i=1}^{r} \sum_{j=1}^{f_i} \sum_{n=1}^{e_i-1} P_{i,j,n}(\alpha)\pi_i^n\alpha_{i,j} = 0
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\[
\sum_{i=1}^{r} \sum_{j=1}^{f_i} \sum_{n=0}^{e_i-1} P_{i,j,n}(\alpha) \pi_i^n \pi_l^{-m} \alpha_{i,j} = 0.
\]
Suppose we have a non-trivial linear combination

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We note that the summands above are in \( \mathcal{M}_l \) for all \( i \neq l \).
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$$\sum_{i=1}^{r} \sum_{j=1}^{f_i} \sum_{n=1}^{e_i-1} P_{i,j,n}(\alpha) \pi_i^n \alpha_{i,j} = 0$$

where $P_{i,j,n}(\alpha) \in \mathbb{F}_p[\alpha]$ for all $i$, $j$, and $n$. As before, we may assume without loss of generality that $\alpha$ doesn’t divide all the $P_{i,j,n}(\alpha)$. We then have indices $l \in \{1, \ldots, r\}$ and $m \in \{1, \ldots, e_l - 1\}$ such that $\alpha | P_{l,j,n}(\alpha)$ for all $n < m$ and any $j \in \{1, \ldots, f_l\}$ and $\alpha \nmid P_{l,j,m}(\alpha)$ for some $j \in \{1, \ldots, f_l\}$.

Multiplying our linear equation through by $\pi_l^{-m}$ yields

$$\sum_{i=1}^{r} \sum_{j=1}^{f_i} \sum_{n=0}^{e_i-1} P_{i,j,n}(\alpha) \pi_i^n \pi_l^{-m} \alpha_{i,j} = 0.$$ 

We note that the summands above are in $\mathcal{M}_l$ for all $i \neq l$. Further, $P_{l,j,n}(\alpha) \pi_l^{n-m} \alpha_{l,j} \in \mathcal{M}_l$ for all $n \neq m$. Hence we have ...
\[ \sum_{j=1}^{f_i} P_{l,j,m}(\alpha) \alpha_{l,j} \in \mathcal{M}_l. \]
\[
\sum_{j=1}^{f_i} P_{l,j,m}(\alpha) \alpha_{l,j} \in \mathcal{M}_l.
\]

But not all \( P_{l,j,m}(\alpha) \in \mathcal{M}_l \) by construction,
$$\sum_{j=1}^{f_i} P_{l,j,m}(\alpha) \alpha_{l,j} \in \mathcal{M}_l.$$ 

But not all $P_{l,j,m}(\alpha) \in \mathcal{M}_l$ by construction, so this yields a non-trivial linear combination
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But not all \( P_{l,j,m}(\alpha) \in \mathcal{M}_l \) by construction, so this yields a non-trivial linear combination (note \( P_{l,j,m}(\alpha) + \mathcal{M}_l \in \mathbb{F}_p \))
\[
\sum_{j=1}^{f_l} P_{l,j,m}(\alpha)\alpha_{l,j} \in M_l.
\]

But not all \( P_{l,j,m}(\alpha) \in M_l \) by construction, so this yields a non-trivial linear combination (note \( P_{l,j,m}(\alpha) + M_l \in \mathbb{F}_p \)) modulo \( M_l \) for \( \alpha_{l,1}, \ldots, \alpha_{l,f_l} \)
\[ \sum_{j=1}^{f_i} P_{l,j,m}(\alpha)\alpha_{l,j} \in \mathcal{M}_l. \]

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\[
\sum_{j=1}^{f_i} P_{l,j,m}(\alpha) \alpha_{l,j} \in \mathcal{M}_l.
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But not all \( P_{l,j,m}(\alpha) \in \mathcal{M}_l \) by construction, so this yields a non-trivial linear combination (note \( P_{l,j,m}(\alpha) + \mathcal{M}_l \in \mathbb{F}_p \)) modulo \( \mathcal{M}_l \) for \( \alpha_{l,1}, \ldots, \alpha_{l,f_i} \) which were supposed to be linearly independent over \( \mathcal{M}_l \). This proves our claim above, whence (5).
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Via (5) we immediately see that \( \deg\left( \text{div}(\alpha)^+ \right) \leq [K : \mathbb{F}_p(\alpha)] \).
\[ \sum_{j=1}^{f_i} P_{l,j,m}(\alpha)\alpha_{l,j} \in \mathcal{M}_l. \]

But not all \( P_{l,j,m}(\alpha) \in \mathcal{M}_l \) by construction, so this yields a non-trivial linear combination (note \( P_{l,j,m}(\alpha) + \mathcal{M}_l \in \mathbb{F}_p \)) modulo \( \mathcal{M}_l \) for \( \alpha_{l,1}, \ldots, \alpha_{l,f_l} \) which were supposed to be linearly independent over \( \mathcal{M}_l \). This proves our claim above, whence (5).

Via (5) we immediately see that \( \deg \left( \text{div}(\alpha^+) \right) \leq [K : \mathbb{F}_p(\alpha)] \). Clearly \( \text{div}(\alpha^-) = \text{div}(\alpha^{-1})^+ \).
\[
\sum_{j=1}^{f_i} P_{l,j,m}(\alpha) \alpha_{l,j} \in \mathcal{M}_l.
\]

But not all \( P_{l,j,m}(\alpha) \in \mathcal{M}_l \) by construction, so this yields a non-trivial linear combination (note \( P_{l,j,m}(\alpha) + \mathcal{M}_l \in \mathbb{F}_p \)) modulo \( \mathcal{M}_l \) for \( \alpha_{l,1}, \ldots, \alpha_{l,f_i} \) which were supposed to be linearly independent over \( \mathcal{M}_l \). This proves our claim above, whence (5).

Via (5) we immediately see that \( \deg \left( \div(\alpha)^+ \right) \leq [K : \mathbb{F}_p(\alpha)] \). Clearly \( \div(\alpha)^- = \div(\alpha^{-1})^+ \). Since \( \mathbb{F}_p(\alpha) = \mathbb{F}_p(\alpha^{-1}) \),

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\[ \sum_{j=1}^{f_l} P_{l,j,m}(\alpha) \alpha_{l,j} \in \mathcal{M}_l. \]

But not all \( P_{l,j,m}(\alpha) \in \mathcal{M}_l \) by construction, so this yields a non-trivial linear combination (note \( P_{l,j,m}(\alpha) + \mathcal{M}_l \in \mathbb{F}_p \)) modulo \( \mathcal{M}_l \) for \( \alpha_{l,1}, \ldots, \alpha_{l,f_l} \) which were supposed to be linearly independent over \( \mathcal{M}_l \). This proves our claim above, whence (5).

Via (5) we immediately see that \( \deg \left( \text{div}(\alpha)^+ \right) \leq [K : \mathbb{F}_p(\alpha)] \). Clearly \( \text{div}(\alpha)^- = \text{div}(\alpha^{-1})^+ \). Since \( \mathbb{F}_p(\alpha) = \mathbb{F}_p(\alpha^{-1}) \), the Proposition follows.
\[
\sum_{j=1}^{f_i} P_{l,j,m}(\alpha)\alpha_{l,j} \in \mathcal{M}_l.
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But not all \( P_{l,j,m}(\alpha) \in \mathcal{M}_l \) by construction, so this yields a non-trivial linear combination (note \( P_{l,j,m}(\alpha) + \mathcal{M}_l \in \mathbb{F}_p \)) modulo \( \mathcal{M}_l \) for \( \alpha_{l,1}, \ldots, \alpha_{l,f_i} \) which were supposed to be linearly independent over \( \mathcal{M}_l \). This proves our claim above, whence (5).

Via (5) we immediately see that \( \deg(\text{div}(\alpha)^+) \leq [K : \mathbb{F}_p(\alpha)] \). Clearly \( \text{div}(\alpha)^- = \text{div}(\alpha^{-1})^+ \). Since \( \mathbb{F}_p(\alpha) = \mathbb{F}_p(\alpha^{-1}) \), the Proposition follows.

The Proposition tells us that we do, indeed, have the analog of principal fractional ideals.
$$\sum_{j=1}^{f_i} P_{l,j,m}(\alpha)\alpha_{l,j} \in \mathcal{M}_l.$$  

But not all $P_{l,j,m}(\alpha) \in \mathcal{M}_l$ by construction, so this yields a non-trivial linear combination (note $P_{l,j,m}(\alpha) + \mathcal{M}_l \in \mathbb{F}_p$) modulo $\mathcal{M}_l$ for $\alpha_{l,1}, \ldots, \alpha_{l,f_i}$ which were supposed to be linearly independent over $\mathcal{M}_l$. This proves our claim above, whence (5).

Via (5) we immediately see that $\deg \left( \text{div}(\alpha)^+ \right) \leq [K : \mathbb{F}_p(\alpha)]$. Clearly $\text{div}(\alpha)^- = \text{div}(\alpha^{-1})^+$. Since $\mathbb{F}_p(\alpha) = \mathbb{F}_p(\alpha^{-1})$, the Proposition follows.

The Proposition tells us that we do, indeed, have the analog of principal fractional ideals.

One readily verifies that the principal divisors are a subgroup of $\text{Div}(K)$,
\[ \sum_{j=1}^{f_i} P_{l,j,m}(\alpha) \alpha_{l,j} \in \mathcal{M}_l. \]

But not all \( P_{l,j,m}(\alpha) \in \mathcal{M}_l \) by construction, so this yields a non-trivial linear combination (note \( P_{l,j,m}(\alpha) + \mathcal{M}_l \in \mathbb{F}_p \)) modulo \( \mathcal{M}_l \) for \( \alpha_{l,1}, \ldots, \alpha_{l,f_i} \) which were supposed to be linearly independent over \( \mathcal{M}_l \).

This proves our claim above, whence (5).

Via (5) we immediately see that \( \deg (\text{div}(\alpha)^+) \leq [K : \mathbb{F}_p(\alpha)] \). Clearly \( \text{div}(\alpha)^- = \text{div}(\alpha^{-1})^+ \). Since \( \mathbb{F}_p(\alpha) = \mathbb{F}_p(\alpha^{-1}) \), the Proposition follows.

The Proposition tells us that we do, indeed, have the analog of principal fractional ideals.

One readily verifies that the principal divisors are a subgroup of \( \text{Div}(K) \), whence we have a factor group and an analog of the class number
\[ \sum_{j=1}^{f_i} P_{l,j,m}(\alpha)\alpha_{l,j} \in \mathcal{M}_l. \]

But not all \( P_{l,j,m}(\alpha) \in \mathcal{M}_l \) by construction, so this yields a non-trivial linear combination (note \( P_{l,j,m}(\alpha) + \mathcal{M}_l \in \mathbb{F}_p \)) modulo \( \mathcal{M}_l \) for \( \alpha_{l,1}, \ldots, \alpha_{l,f_i} \) which were supposed to be linearly independent over \( \mathcal{M}_l \). This proves our claim above, whence (5).

Via (5) we immediately see that \( \deg \left( \text{div}(\alpha)^+ \right) \leq [K : \mathbb{F}_p(\alpha)] \). Clearly \( \text{div}(\alpha^-) = \text{div}(\alpha^{-1})^+ \). Since \( \mathbb{F}_p(\alpha) = \mathbb{F}_p(\alpha^{-1}) \), the Proposition follows.

The Proposition tells us that we do, indeed, have the analog of principal fractional ideals.

One readily verifies that the principal divisors are a subgroup of \( \text{Div}(K) \), whence we have a factor group and an analog of the class number (which we haven’t yet proven to be finite).