1. Complete the proof of Theorem 1 of the handout on applications of Minkowski’s Theorem.

2. Prove Corollary 1 to Theorem 2 of the handout on applications of Minkowski’s Theorem by applying that theorem to the case $\mathfrak{A} = \mathcal{O}_K$. Use the fact that $|N_{K/\mathbb{Q}}(\alpha)| \geq 1$ for all non-zero $\alpha \in \mathcal{O}_K$ and show that $\frac{n!}{n^n}(4/\pi)^{r_2} < 1$ whenever $n = [K : \mathbb{Q}] > 1$.

3. Let $\mathcal{O}_R$ be an integral domain with quotient field $K$. A multiplicative subset of $\mathcal{O}_R$ is a subset $S \subset \mathcal{O}_R$ containing 1 but not 0 and such that $ab \in S$ whenever $a, b \in S$. If $S$ is a multiplicative subset of $\mathcal{O}_R$, prove that $S^{-1}\mathcal{O}_R := \{a/b \in K : a \in \mathcal{O}_R, b \in S\}$ is a subring of $K$ and that the set theoretic complement $\mathcal{O}_R \setminus P$ of a prime ideal $P \subset \mathcal{O}_R$ is a multiplicative subset. Explicitly describe $S^{-1}\mathcal{O}_R$ for the case where $R = \mathbb{Z}$ and $S = \mathbb{Z} \setminus p\mathbb{Z}$ for a prime $p$.

4. Let $K$ be a number field and let $\mathfrak{P}$ be a non-zero prime ideal of $\mathcal{O}_K$. By #3 above $S = \mathcal{O}_K \setminus \mathfrak{P}$ is a multiplicative subset of $\mathcal{O}_K$; to ease notation, set $\mathcal{O}_{\mathfrak{P}} = S^{-1}\mathcal{O}_K$.
   a) Show that $\mathcal{O}_{\mathfrak{P}}$ has a unique maximal ideal. Indeed, the non-units here form an ideal.
   b) Write $\mathfrak{M}_{\mathfrak{P}}$ for the unique maximal ideal of $\mathcal{O}_{\mathfrak{P}}$. Show that $\mathfrak{M}_{\mathfrak{P}}$ is a principal ideal and describe the set of $\alpha/\beta \in \mathfrak{M}_{\mathfrak{P}}$ that generate $\mathfrak{M}_{\mathfrak{P}}$. Prove that all ideals of $\mathcal{O}_{\mathfrak{P}}$ besides the zero ideal and the ring itself are of the form $\mathfrak{M}_{\mathfrak{P}}^e$ for some positive integer $e$, whence $\mathcal{O}_{\mathfrak{P}}$ is a principal ideal domain. Hint: it may be helpful to use the notion of order at $\mathfrak{P}$ of a non-zero $\alpha \in K$. 

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