Absolute Values on \( \mathbb{Q} \)

Suppose \( p \) is a prime number. By the Fundamental Theorem of Arithmetic (unique factorization of integers), every non-zero integer \( n \) can be uniquely written in the form \( n = p^k m \) where \( p \nmid m \). This unique power \( k \) is called the order of \( n \) at \( p \) and denoted \( \text{ord}_p(n) \). By convention, \( \text{ord}_p(0) = \infty \).

**Definition:** Suppose \( r/s \in \mathbb{Q} \), where \( r, s \in \mathbb{Z} \). Then \( \text{ord}_p(r/s) = \text{ord}_p(r) - \text{ord}_p(s) \).

Note how this definition really is ... a definition. In other words, it doesn’t depend on how you write the rational number. For instance, \( \text{ord}_p(1/2) \) really is the same as \( \text{ord}_p(2/4) \) and \( \text{ord}_p(17/34) \).

This actually follows from the integer case of the following lemma.

**Lemma 0:** Suppose \( p \) is a prime and \( a, b \in \mathbb{Q} \). Then

\[
\text{ord}_p(ab) = \text{ord}_p(a) + \text{ord}_p(b).
\]

Also,

\[
\text{ord}_p(a + b) \geq \min\{\text{ord}_p(a), \text{ord}_p(b)\},
\]

with equality if \( \text{ord}_p(a) \neq \text{ord}_p(b) \).

**Proof:** This is obvious if either \( a \) or \( b \) is zero, so we will assume \( ab \neq 0 \).

First suppose \( a, b \in \mathbb{Z} \). Write \( a = p^{\text{ord}_p(a)} m_1 \) and \( b = p^{\text{ord}_p(b)} m_2 \), where \( p \nmid m_1 \) and \( p \nmid m_2 \). Then \( ab = p^{\text{ord}_p(a)+\text{ord}_p(b)} m_1 m_2 \) where \( p \nmid m_1 m_2 \), which proves the first statement.

Without loss of generality, \( \text{ord}_p(a) \leq \text{ord}_p(b) \). We then have

\[
a + b = p^{\text{ord}_p(a)} (m_1 + p^{\text{ord}_p(b) - \text{ord}_p(a)} m_2),
\]

and clearly \( p \nmid (m_1 + p^{\text{ord}_p(b) - \text{ord}_p(a)} m_2) \) if \( \text{ord}_p(b) > \text{ord}_p(a) \). Thus, the lemma is true for integers \( a \) and \( b \).

Now suppose \( a, b \in \mathbb{Q} \) and write \( a = r/s, b = u/v \) for \( r, s, u, v \in \mathbb{Z} \). Then \( ab = (ru/sv) \) and by what we’ve already shown

\[
\text{ord}_p(ab) = \text{ord}_p(ru) - \text{ord}_p(sv) = \text{ord}_p(r) + \text{ord}_p(u) - \text{ord}_p(s) - \text{ord}_p(v) = \text{ord}_p(a) + \text{ord}_p(b).
\]
Further,

$$\text{ord}_p(sv(a + b)) = \text{ord}_p(vr + su) \geq \min\{\text{ord}_p(vr), \text{ord}_p(su)\}$$

$$= \min\{\text{ord}_p(v) + \text{ord}_p(r), \text{ord}_p(s) + \text{ord}_p(u)\}$$

$$= (\text{ord}_p(s) + \text{ord}_p(v)) \min\{\text{ord}_p(r) - \text{ord}_p(s), \text{ord}_p(u) - \text{ord}_p(v)\}$$

$$= (\text{ord}_p(sv)) \min\{\text{ord}_p(a), \text{ord}_p(b)\},$$

with equality if \(\text{ord}_p(vr) \neq \text{ord}_p(su)\). By what we have already shown, this implies that \(\text{ord}_p(a + b) \geq \min\{\text{ord}_p(a), \text{ord}_p(b)\}\), with equality if \(\text{ord}_p(v) + \text{ord}_p(r) \neq \text{ord}_p(s) + \text{ord}_p(u)\). Thus, \(\text{ord}_p(a + b) = \min\{\text{ord}_p(a), \text{ord}_p(b)\}\) if \(\text{ord}_p(r) - \text{ord}_p(s) \neq \text{ord}_p(u) - \text{ord}_p(v)\), i.e., if \(\text{ord}_p(a) \neq \text{ord}_p(b)\).

So the behavior of the \(\text{ord}_p\) function is somewhat reminiscent of the degree function on polynomials. In fact, it behaves just like \textit{minus} the degree function except the degree is a function on polynomials whereas \(\text{ord}_p\) is a function on \(\mathbb{Q}\).

**Definition:** If \(p\) is a prime, then the \(p\)-adic absolute value \(|\cdot|_p\) on \(\mathbb{Q}\) is defined to be

$$|a|_p = p^{-\text{ord}_p(a)}$$

for all \(a \in \mathbb{Q}\), with the usual convention that \(\text{“}p^{-\infty}\text{”} = 0\).

Technically we’re getting ahead of ourselves here. We should first say what an absolute value is in the first place.

**Definition:** A function \(|\cdot|\) from \(\mathbb{Q}\) to itself is called an absolute value if it satisfies the following:

i) \(|a| \geq 0\), with equality if and only if \(a = 0\),

ii) \(|ab| = |a||b|\) for all \(a, b \in \mathbb{Q}\),

iii) \(|a + b| \leq |a| + |b|\).

The last condition here is called the triangle inequality. If the stronger condition

iv) \(|a + b| \leq \max\{|a|, |b|\}\)

holds, we say the absolute value is non-archimedean. If not, the absolute value is called archimedean.

Typically one allows the absolute value to take non-negative real values. There really isn’t any loss of generality here, though (see the comment before Lemma 8). Our definition also has the advantage of not assuming the existence of \(\mathbb{R}\), an advantage which will be apparent very soon.
Examples: The usual absolute value, which we’ll denote \(|·|_∞\), is an archimedean absolute value on \(\mathbb{Q}\). For any prime number \(p\), the \(p\)-adic absolute value defined by \(|a|_p = p^{-\text{ord}_p(a)}\) is a non-archimedean absolute value.

It turns out (Ostrowski’s Theorem) that the above examples are essentially all the absolute values on \(\mathbb{Q}\). (Here “essentially” refers to the topologies produced.) For this reason, one often refers to the “prime at infinity” as giving the usual archimedean absolute value. We’ll use \(v\) to denote either a prime number or the prime at infinity; a \(v\)-adic absolute value is either \(|·|_∞\) or one of the \(p\)-adic absolute values above.

Definition: Let \(|·|\) be an absolute value on \(\mathbb{Q}\). A Cauchy sequence is a sequence \(\{a_n\}\) of rational numbers such that, for all \(ε > 0\), there is an \(N \in \mathbb{N}\) where \(|a_n - a_m| < ε\) for all \(n, m \geq N\).

Examples:
1) For a fixed \(c \in \mathbb{Q}\), the constant sequence \(a_n = c\) for all \(n\) is a Cauchy sequence for all \(|·|_v\). The special case where \(c = 0\) is called the null sequence.
2) The sequence given by \(a_n = \frac{10^n - 1}{10^n}\) is a Cauchy sequence for the usual absolute value, but not for any of the \(p\)-adic absolute values.
3) The sequence given by \(a_n = 2^n\) is a Cauchy sequence for the 2-adic absolute value, but not for any other \(v\)-adic absolute value.

Lemma 1: Suppose \(\{a_n\}\) and \(\{b_n\}\) are Cauchy sequences. Then so are the sequences given by \(c_n = a_n + b_n\) and \(d_n = a_nb_n\).

Proof: Let \(ε > 0\) and let \(M, N \in \mathbb{N}\) with \(|a_m - a_n| < ε/2\) for all \(m, n \geq M\) and \(|b_m - b_n| < ε/2\) for all \(m, n \geq N\). Then for all \(m, n \geq \max\{M, N\}\) we have

\[|c_m - c_n| = |a_m - a_n + b_m - b_n| \leq |a_m - a_n| + |b_m - b_n| < ε.\]

Let \(M_0, N_0 \in \mathbb{N}\) with \(|a_m - a_n| < 1\) for all \(m, n \geq M_0\) and \(|b_m - b_n| < 1\) for all \(m, n \geq N_0\). Then by the triangle inequality \(|a_m| < |a_{M_0}| + 1\) for all \(m \geq M_0\) and \(|b_m| \leq |b_{N_0}| + 1\) for all \(m \geq N_0\). Let \(M, N \in \mathbb{N}\) with \(|a_m - a_n| < ε/2(|a_{M_0}| + 1)\) for all \(m, n \geq M\) and \(|b_m - b_n| < ε/2(|b_{N_0}| + 1)\) for all \(m, n \geq N\). Then for all \(m, n \geq \max\{M_0, N_0, M, N\}\) we have

\[|d_m - d_n| = |(a_m - a_n)b_m + (b_m - b_n)a_n| \leq |a_m - a_n||b_m| + |b_m - b_n||a_n| < ε.\]

Definition: Let \(|·|\) be an absolute value on \(\mathbb{Q}\) and let \(c \in \mathbb{Q}\). A sequence \(\{a_n\}\) is said to
converge to $c$ in the absolute value $| \cdot |$, and we write $a_n \rightarrow c$, if for every $\varepsilon > 0$ there is an $N \in \mathbb{N}$ such that $|a_n - c| < \varepsilon$ for all $n \geq N$.

**Lemma 2:** If a sequence $\{a_n\}$ converges in an absolute value, then it is a Cauchy sequence for that absolute value.

**Proof:** Suppose $a_n \rightarrow c$ and let $\varepsilon > 0$. Get an $N \in \mathbb{N}$ such that $|a_n - c| < \varepsilon/2$ for all $n \geq N$. Then by the triangle inequality

$$|a_n - a_m| = |(a_n - c) - (a_m - c)| \leq |a_n - c| + |a_m - c| < \varepsilon$$

for all $n, m \geq N$.

In the examples of Cauchy sequences above, $\frac{10^n - 1}{10^n} \rightarrow 1$ in the usual absolute value and $2^n \rightarrow 0$ in the 2-adic absolute value. Such Cauchy sequences are actually relatively uncommon, it turns out. Specifically, “most” Cauchy sequences don’t converge.

**Definition:** Two Cauchy sequences $\{a_n\}$ and $\{b_n\}$ of rational numbers are called equivalent, and we write $\{a_n\} \sim \{b_n\}$, if $a_n - b_n \rightarrow 0$.

**Lemma 3:** This is an equivalence relation.

**Proof:** For any sequence $\{a_n\}$ of rational numbers, $a_n - a_n \rightarrow 0$, so that $\{a_n\} \sim \{a_n\}$ for any Cauchy sequence $\{a_n\}$.

Suppose $\{a_n\} \sim \{b_n\}$ and let $\varepsilon > 0$. Then for some $N \in \mathbb{N}$, $|a_n - b_n| < \varepsilon$ for all $n \geq N$. Thus $|b_n - a_n| < \varepsilon$ for $n \geq N$ and $b_n - a_n \rightarrow 0$. (Note that $| - 1| = 1$ from part ii of the definition of absolute value.) In other words, $\{b_n\} \sim \{a_n\}$.

Suppose $\{a_n\} \sim \{b_n\}$ and $\{b_n\} \sim \{c_n\}$ and let $\varepsilon > 0$. Then for some $N \in \mathbb{N}$, $|a_n - b_n| < \varepsilon/2$ for all $n \geq N$ and for some $M \in \mathbb{N}$, $|b_n - c_n| < \varepsilon/2$ for all $n \geq M$. This implies that

$$|a_n - c_n| = |a_n - b_n + b_n - c_n| \leq |a_n - b_n| + |b_n - c_n| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

for all $n \geq \max\{N, M\}$. Thus $\{a_n - c_n\} \rightarrow 0$ and $\{a_n\} \sim \{c_n\}$.

**Definition/Notation:** The real numbers $\mathbb{R}$ is the set of equivalence classes of Cauchy sequences of rational numbers for the archimedean absolute value; we also write $\mathbb{Q}_\infty$ for the real numbers. The $p$-adic numbers $\mathbb{Q}_p$ is the set of equivalence classes of Cauchy sequences for the $p$-adic absolute value. For a Cauchy sequence $\{a_n\}$ we’ll write $[\{a_n\}]$ to denote the equivalence
class of Cauchy sequences \{b_n\} with \{a_n\} \sim \{b_n\}. We view \mathbb{Q} as a subset of \mathbb{Q}_v by identifying the rational number \(c\) with the sequence given by \(a_n = c\) for all \(n\).

Definition: Define addition and multiplication in \(\mathbb{Q}_v\) by

\[[\{a_n\}] + [\{b_n\}] = [\{a_n + b_n\}]\]

and

\[[\{a_n\}] \cdot [\{b_n\}] = [\{a_n \cdot b_n\}]\]

Lemma 4: These operations are well-defined, i.e., depend only on the equivalence classes and not the particular elements of the equivalence classes used.

Proof: Suppose \{a_n\} \sim \{a'_n\} and \{b_n\} \sim \{b'_n\} are all Cauchy sequences. By Lemma 2, both \{a_n + b_n\} and \{a_n b_n\} are Cauchy sequences.

Let \(\epsilon > 0\). Then for some \(N_1, M_1 \in \mathbb{N}\), \(|a_n - a'_n| < \epsilon/2\) for all \(n \geq N_1\) and \(|b_n - b'_n| < \epsilon/2\) for all \(n \geq M_1\). This implies that

\[|(a_n + b_n) - (a'_n + b'_n)| = |(a_n - a'_n) + (b_n - b'_n)| \leq |a_n - a'_n| + |b_n - b'_n| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon\]

for all \(n \geq \max\{N_1, M_1\}\). Thus, \(a_n + b_n \sim a'_n + b'_n\) and addition is well-defined.

By the proof of Lemma 2, all four sequences \{a_n\}, \{a'_n\}, \{b_n\} and \{b'_n\} are bounded. Thus, there is a \(B \geq 1\) such that \(|a_n|, |a'_n|, |b_n|, |b'_n| \leq B\) for all \(n \in \mathbb{N}\). For some \(N_2, M_2 \in \mathbb{N}\) we have \(|a_n - a'_n| < \frac{\epsilon}{2B}\) for all \(n \geq N_2\) and \(|b_n - b'_n| < \frac{\epsilon}{2B}\) for all \(n \geq M_2\). This implies that

\[|a_n b_n - a'_n b'_n| = |(a_n - a'_n)b_n + a'_n(b_n - b'_n)|\]

\[\leq |a_n - a'_n||b_n| + |a'_n||b_n - b'_n|\]

\[< \frac{\epsilon}{2B} \cdot B + \frac{\epsilon}{2B} \cdot B\]

\[= \epsilon\]

for all \(n \geq \max\{N_2, M_2\}\). Thus, \(|a_n b_n - a'_n b'_n| < \epsilon\) for all \(n \geq \max\{N_2, M_2\}\) and \(a_n b_n \sim a'_n b'_n\).

This shows that multiplication is well-defined.

Theorem 1: For any \(v\), \(\mathbb{Q}_v\) is a field containing (an isomorphic copy of) \(\mathbb{Q}\).

Proof: The only thing not immediate here is the existence of multiplicative inverses. Suppose that \([\{a_n\}]\) is not equivalent to the null sequence. Then for some \(\epsilon > 0\) there are infinitely many
n \in \mathbb{N} \text{ such that } |a_n| \geq \epsilon. \text{ Since } \{a_n\} \text{ is a Cauchy sequence, there is an } N \in \mathbb{N} \text{ such that } |a_n - a_m| < \epsilon/2 \text{ for all } n, m \geq N. \text{ Since there must be an } n_0 \geq N \text{ such that } |a_{n_0}| \geq \epsilon, \text{ we have}

|a_m| \geq |a_{n_0}| - |a_{n_0} - a_m| > \epsilon - \frac{\epsilon}{2} = \frac{\epsilon}{2}

for all } m \geq N. \text{ In particular, } a_m \neq 0 \text{ for } m \geq N. \text{ Further, it is not difficult to see that the sequence } \{b_n\} \text{ defined by}

\[
b_n = \begin{cases} a_N & \text{if } n \leq N, \\ a_n & \text{if } n \geq N \end{cases}
\]

is a Cauchy sequence equivalent to } \{a_n\}. \text{ Thus, we may assume without loss of generality that } a_n \neq 0 \text{ for all } n \in \mathbb{N}. \text{ Then easily } \{a_n^{-1}\} \text{ is a multiplicative inverse for } \{a_n\}.

Next, we extend our } v\text{-adic absolute values to } \mathbb{Q}_v. 

**Definition:** For } \{a_n\} \in \mathbb{Q}_v, \text{ set } \|\{a_n\}\|_v = \|\{a_n\}\| \in \mathbb{R} 

**Lemma 5:** This is well-defined, i.e., } \{a_n\}_v \text{ is a Cauchy sequence for the archimedean absolute value and } \{a_n\}_v \sim \{a'_n\}_v \text{ whenever } \{a_n\} \text{ and } \{a'_n\} \text{ are equivalent Cauchy sequences for the } v\text{-adic absolute value.} 

**Proof:** Let } \epsilon > 0. \text{ Then for some } N \in \mathbb{N} \text{ we have } |a_n - a_m|_v < \epsilon \text{ for all } n, m \geq N. \text{ By the triangle inequality, } |a_n - a_m|_v \leq |a_n - a_m|_v \text{ and } |a_m - a_n|_v \leq |a_m - a_n|_v. \text{ As remarked above, } 

|a_n - a_m|_v = |a_m - a_n|_v, \text{ so that } \|a_n|_v - |a_m|_v \|_\infty \leq |a_n - a_m|_v. \text{ This implies that } \|a_n|_v - |a_m|_v \|_\infty < \epsilon \text{ for all } n, m \geq N \text{ and } \{a_n\}_v \text{ is a Cauchy sequence for the archimedean absolute value.} 

Suppose } \{a'_n\} \sim \{a_n\} \text{ and let } \epsilon > 0. \text{ Then for some } N \in \mathbb{N}, \ |a'_n - a_n|_v < \epsilon \text{ for all } n \geq N. \text{ As above, this implies that } \|a'_n|_v - |a_n|_v \|_\infty < \epsilon \text{ for all } n \geq N, \text{ so that } \{a'_n\}_v \text{ and } \{a_n\}_v \text{ are equivalent Cauchy sequences for the archimedean absolute value.} 

In order to show these extensions are “absolute values,” we first need to discuss the archimedean case.

**Definition:** We say a real number } a \text{ is greater than zero (or positive) and write } a > 0 \text{ if there is an } N \in \mathbb{N} \text{ and an } \epsilon > 0 \text{ such that } a_n \geq \epsilon \text{ for all } n \geq N, \text{ where } \{a_n\} \text{ is a Cauchy sequence in the equivalence class } a.

**Lemma 6:** This is well-defined.

**Proof:** Suppose } \{a'_n\} \text{ and } \{a_n\} \text{ are equivalent Cauchy sequences for the archimedean absolute value. Suppose further that there is some } N \in \mathbb{N} \text{ and an } \epsilon > 0 \text{ such that } a_n \geq \epsilon \text{ for all } n \geq N. \text{ There
is an \( M \in \mathbb{N} \) such that \(|a'_n - a_n| < \epsilon/2\) for all \( n \geq M \). This implies that \(|a'_n| \geq |a_n| - |a'_n - a_n| > \epsilon - \frac{\epsilon}{2} = \frac{\epsilon}{2}\) for all \( n \geq \max\{N, M\}\).

**Definition:** We say a real number \( a \) is greater than a real number \( b \), and write \( a > b \), if \( a - b > 0 \).

Suppose \( a = \{a_n\} \) and \( b = \{b_n\} \), where \( \{a_n\} \) and \( \{b_n\} \) are Cauchy sequences for the archimedean absolute value. Then \( a > b \) means there is an \( \varepsilon > 0 \) and \( N \in \mathbb{N} \) with \( a_n \geq b_n + \varepsilon \) for all \( n \geq N \). This implies that \( a_n > b_n \) for all \( n \geq N \). Note, however, that \( a_n > b_n \) for all \( n \) greater than or equal to some \( N \in \mathbb{N} \) does not imply that \( a > b \) (though it does imply that \( a \geq b \)).

**Lemma 7:** The real numbers are totally ordered by \(|\cdot|\). In other words, for \( a, b \in \mathbb{R} \) with \( a \neq b \), either \( a > b \) or \( b > a \).

**Proof:** Write \( a = \{a_n\} \) and \( b = \{b_n\} \) as above and suppose that \( a \not\geq b \) and \( b \not\geq a \). Let \( \varepsilon > 0 \). By Lemma 2, \( \{a_n - b_n\} \) is a Cauchy sequence, so there is an \( N \in \mathbb{N} \) such that \(|(a_m - b_m) - (a_n - b_n)| < \varepsilon\) for all \( m, n \geq N \). Suppose \( m \geq N \). If \( a_m - b_m \geq 0 \), then since \( a \not\geq b \) there are infinitely many \( n \geq N \) with \( a_n - b_n \leq 0 \). For such an \( n \), we see by the above inequality that \( a_m - b_m < \varepsilon \). Similarly, if \( a_m - b_m \leq 0 \), using \( b \not\geq a \) we get \( a_m - b_m > -\varepsilon \). Thus, \(|a_m - b_m| < \varepsilon\) for all \( m \geq N \) and \( a - b = 0 \).

**Theorem 2:** As defined above, for any \( v \) the function \(|\cdot|_v\) on \( \mathbb{Q}_v \) is an absolute value. In other words, the following three properties hold:

1. \(|a|_v \geq 0\), with equality if and only if \( a = 0 \) for all \( a \in \mathbb{Q}_v \),
2. \(|ab|_v = |a|_v|b|_v\) for all \( a, b \in \mathbb{Q}_v \),
3. \(|a + b|_v \leq |a|_v + |b|_v\) for all \( a, b \in \mathbb{Q}_v \).

Moreover, we have

4. \(|a + b|_v \leq \max\{|a|_v, |b|_v\}\) for all \( a, b \in \mathbb{Q}_v \)

if \( v \neq \infty \). That is, \(|\cdot|_v\) is non-archimedean for \( v \neq \infty \).

**Proof:** Clearly \(|a|_v \geq 0\) and \(|0|_v = 0\) (though one should be careful here as the first “0” is an element of \( \mathbb{Q}_v \) while the second is an element of \( \mathbb{R} = \mathbb{Q}_\infty \)). Suppose that \( a \neq 0 \). Writing \( a = \{a_n\} \) where \( \{a_n\} \) is a Cauchy sequence for the \( v \)-adic absolute value, there must be an \( \epsilon > 0 \) such that \(|a_n|_v \geq \epsilon\) for infinitely many \( n \in \mathbb{N} \). Let \( N \in \mathbb{N} \) be such that \(|a_n - a_m|_v < \epsilon/2\) for all \( n, m \geq N \). Since there is an \( n_0 \geq N \) with \(|a_{n_0}|_v \geq \epsilon\), we have \(|a_m|_v \geq |a_{n_0}|_v - |a_{n_0} - a_m|_v > \epsilon - \epsilon/2 = \epsilon/2\) for all \( m \geq n_0 \).
for all \( m \geq N \) by the triangle inequality. Thus, the sequence \( \{a_n\}_v \) is not equivalent to the null sequence, i.e., \( |a|_v \not\to 0 \). This proves part i.

Let \( a, b \in \mathbb{Q}_v \), and write \( a = \{a_n\} \) and \( b = \{b_n\} \) for Cauchy sequences \( \{a_n\} \) and \( \{b_n\} \). Using \( |a_n b_n|_v = |a_n|_v \cdot |b_n|_v \) for all \( n \), we have

\[
|\{a_n\}|_v \cdot |\{b_n\}|_v = |\{a_n b_n\}|_v = |\{a_n|_v \cdot |\{b_n\}|_v | = |\{a_n\}|_v \cdot |\{b_n\}|_v.
\]

This proves part ii. By the triangle inequality, \( |a_n + b_n|_v - (|a_n|_v + |b_n|_v) \leq 0 \) for all \( n \). By the discussion above, this implies that \( |\{a_n + b_n\}|_v \not\to \{\{a_n\} + \{b_n\}\} \). Part iii follows from this and Lemma 7. The proof of part iv is similar.

**Definition:** A sequence \( \{a_n\} \subset \mathbb{Q}_v \) is called Cauchy if for every \( \varepsilon > 0 \) there is an \( N \in \mathbb{N} \) such that \( |a_n - a_m|_v < \varepsilon \) for all \( m, n \geq N \).

Technically speaking, the \( \varepsilon \) here is allowed to be any positive real number. The next lemma shows that it suffices (if one is so inclined) to only use positive rational \( \varepsilon \).

**Lemma 8:** Suppose \( a, b \in \mathbb{Q}_v \). If \( a \not= b \), then there is a rational number \( c \) such that \( |a - c|_v < |a - b|_v \).

**Proof:** As above, write \( a = \{a_n\} \) and \( b = \{b_n\} \). Since \( \{a_n\} \not\sim \{b_n\} \), \( \{a_n - b_n\} \) is not equivalent to the null sequence. By Theorem 2, \( |\{a_n\} - \{b_n\}|_v = |\{a_n - b_n\}|_v > 0 \) so that there is an \( \varepsilon > 0 \) and an \( N \in \mathbb{N} \) such that \( |a_n - b_n|_v \geq \varepsilon \) for all \( n \geq N \). Also, for some \( M \in \mathbb{N} \) we have \( |a_n - a_m|_v < \varepsilon/2 \) for all \( n, m \geq M \). Let \( c = a_{N+M} \). Then for all \( n \geq N + M \), \( |a_n - c|_v < \varepsilon/2 \) and \( |a_n - b_n|_v \geq \varepsilon \). In particular, \( |a_n - b_n|_v - |a_n - c|_v \geq \varepsilon/2 \) for all \( n \geq N + M \). By the definitions, this means that \( |a - c|_v < |b - c|_v \).

**Theorem 3:** For all \( v \), the field \( \mathbb{Q}_v \) is topologically complete with respect to the \( v \)-adic absolute value, i.e., every Cauchy sequence in \( \mathbb{Q}_v \) converges.

**Proof:** Let \( \{r_n\} \) be a Cauchy sequence in \( \mathbb{Q}_v \). If there is an \( r \in \mathbb{Q}_v \) and an \( N \in \mathbb{N} \) such that \( r_n = r \) for all \( n \geq N \) we are done, since in this case \( r_n \to r \). So suppose this is not the case. For each \( n \in \mathbb{N} \) let \( n' \in \mathbb{N} \) be least such that \( n' > n \) and \( r_n \not= r_{n'} \). For each \( n \in \mathbb{N} \) choose (via Lemma 8) an \( a_n \in \mathbb{Q} \) such that \( |r_n - a_n|_v < |r_n - r_{n'}|_v \).

Let \( \varepsilon > 0 \). Then there is an \( N \in \mathbb{N} \) such that \( |r_n - r_m|_v < \varepsilon/3 \) for all \( n, m \geq N \). By the
triangle inequality, we have

\[ |a_n - a_m|_v = |a_n - r_n + r_n - r_m + r_m - a_m|_v \]
\[ \leq |a_n - r_n|_v + |r_n - r_m|_v + |r_m - a_m|_v \]
\[ < |r_n - r_n'|_v + |r_n - r_m|_v + |r_m - r_m'|_v \]
\[ < \epsilon \]

for all \( n, m \geq N \). This shows that \( \{a_n\} \) is a Cauchy sequence (of rational numbers), i.e., \( \{a_n\} \in \mathbb{Q}_v \).

Let \( \epsilon > 0 \) again. Then there are is an \( N \in \mathbb{N} \) such that \( |r_n - r_m|_v, |a_n - a_m|_v < \epsilon/3 \) for all \( n, m \geq N \). In particular, \( |r_N - a_N|_v < \epsilon/3 \) and also \( |a_N - \{a_m\}|_v \leq \epsilon/3 \). (Note that \( a_N \) is viewed as an element of \( \mathbb{Q}_v \) here.) Using the triangle inequality once more,

\[ |r_n - \{a_m\}|_v = |r_n - r_N + r_N - a_N + a_N - \{a_m\}|_v \]
\[ \leq |r_n - r_N|_v + |r_N - a_N|_v + |a_N - \{a_m\}|_v \]
\[ < \epsilon \]

for all \( n \geq N \). Thus, \( r_n \to \{a_m\} \in \mathbb{Q}_v \).

Examples:

1. If \( a_n \in \{0, 1, \ldots, 9\} \) for all \( n \), then the partial sums for the series \( \sum_{n=1}^{\infty} a_n 10^{-n} \) form a Cauchy sequence for the archimedean absolute value, so this sum is a (non-negative) real number. In particular, notice that setting \( a_n = 9 \) for all \( n \) gives us a sequence which is equivalent to the sequence of all 1’s. In other words, “0.\overline{9} = 1.” This is just a particular case of the following fact: the decimal representation of a real number is not unique.

2. Similar to above, fix a prime \( p \) and suppose \( a_n \in \{0, 1, \ldots, p-1\} \) for all \( n \). Then the partial sums of \( \sum_{n=1}^{\infty} a_n p^n \) form a Cauchy sequence for the \( p \)-adic absolute value, so this sum is an element of \( \mathbb{Q}_p \).

Given Theorem 3, one can then proceed to do “calculus” with \( \mathbb{Q}_v \). Be forewarned that this gets a little funky for the non-archimedean fields. One can prove the usual results regarding \( > \) and real numbers; this is what really sets \( \mathbb{R} \) apart from the \( p \)-adic numbers \( \mathbb{Q}_p \).

We know that \( X^2 - 2 \) has a root (two, in fact) in \( \mathbb{R} \). What are these roots? Technically speaking, they’re equivalence classes of Cauchy sequences of rational numbers. How could one produce such a Cauchy sequence? Newton’s method typically works quite well in situations such
as this. Indeed, in this case if we set $a_1 = 1$ and recursively define

$$a_{n+1} = a_n + \frac{2 - a_n^2}{2a_n}$$

we get a Cauchy sequence of rational numbers. As a Cauchy sequence in $\mathbb{R}$, it converges to $\sqrt{2}$.

The same type of construction can sometimes work for $p$-adic roots as well; that’s the gist of Hensel’s Lemma.