The Adele Ring

We adopt the standard notation of \( \mathbb{Q} \) for the field of rational numbers and \( \mathbb{F}_p \) for the finite field with \( p \) elements. (We will assume \( p \) is a prime.) Here \( K \) will denote either a number field (finite algebraic extension of \( \mathbb{Q} \)) or function field over a finite field (finite algebraic extension of \( \mathbb{F}_p(T) \) where \( T \) is transcendental over \( \mathbb{F}_p \)). We will assume some familiarity with the notion of the places of such a field; the set of all places of \( K \) will be denoted by \( M(K) \). The place of \( \mathbb{Q} \) corresponding to the usual Euclidean absolute value will be denoted \( \infty \).

For any place \( v \in M(K) \) we can construct the topological completion \( K_v \) of \( K \) via the usual Cauchy sequence stuff. If \( v \mid \infty \) then \( K_v \) is either the field of real numbers \( \mathbb{R} \) or the field of complex numbers \( \mathbb{C} \). When \( p \in M(\mathbb{Q}) \) is a finite place of \( \mathbb{Q} \) (identified with a positive prime in the standard manner) then \( \mathbb{Q}_p \) is the field of \( p \)-adic numbers with subring \( \mathbb{Z}_p \), the \( p \)-adic integers. Set-theoretically, \( \mathbb{Z}_p \) consists of those \( \alpha \in \mathbb{Q}_p \) with absolute value \( |\alpha|_p \leq 1 \), where \( | \cdot |_p \) denotes the \( p \)-adic absolute value on \( \mathbb{Q} \) extended to \( \mathbb{Q}_p \). (Note that the particular normalization, i.e., representative absolute value of the place chosen, is not an issue here.) In general, for any finite place \( v \in M(K) \), meaning \( v \nmid \infty \), we have the maximal compact subring \( \mathfrak{O}_v \subset K_v \) consisting of those elements \( \alpha \in K_v \) with \( |\alpha|_v \leq 1 \).

Suppose \( P \subset M(K) \) is a finite set of places containing all the infinite places of \( K \). Set

\[
K_\mathfrak{a}(P) = \prod_{v \in P} K_v \times \prod_{v \notin P} \mathfrak{O}_v.
\]

**Exercise 1:** Show that \( K_\mathfrak{a}(P) \) is a ring when we define addition and multiplication component-wise, and this makes \( K_\mathfrak{a}(P) \) a topological ring via the topologies on the \( K_v \) induced by the absolute values (the choices in the various places here are irrelevant by the definition of place).

**Lemma 1:** \( K_v \) is always locally compact and \( \mathfrak{O}_v \) is compact when \( v \nmid \infty \). Therefore \( K_\mathfrak{a}(P) \) is locally compact with the usual product topology for any finite subset \( P \subset M(K) \) containing all the infinite places of \( K \).

**Proof:** If \( v \) is an infinite place, then \( K_v \) is \( \mathbb{R} \) or \( \mathbb{C} \), both of which are certainly locally compact (the closure of an open ball works nicely). If \( v \mid \infty \) the local compactness follows immediately from \( \mathfrak{O}_v \) being compact. Now \( \mathfrak{O}_v \) is clearly complete, since any cauchy sequence in \( \mathfrak{O}_v \subset K_v \) converges by the definition of \( K_v \) and must converge to a point in \( \mathfrak{O}_v \) by the ultra-metric inequality. Certainly \( \mathfrak{O}_v \) is totally bounded by its very definition. Thus \( \mathfrak{O}_v \) is compact since it’s a complete and totally bounded metric space.

**Definition:** The adele ring \( K_\mathfrak{a} \) is defined to be the union of all \( K_\mathfrak{a}(P) \), where the union is taken over all finite subsets \( P \subset M(K) \) where \( P \) contains all finite places of \( K \).

**Exercise 2:** Show that set-theoretically \( K_\mathfrak{a} \) consists of ordered tuples \((\alpha_v)_{v \in M(K)} \) where \( \alpha_v \in K_v \) for all \( v \in M(K) \) and \( \alpha_v \in \mathfrak{O}_v \) for all but finitely many \( v \in M(K) \). Prove that \( K_\mathfrak{a} \) is a ring with subrings \( K_\mathfrak{a}(P) \) for all finite subsets \( P \subset M(K) \) containing all infinite places of \( K \).

We put a topology on \( K_\mathfrak{a} \) (sometimes called the restricted direct product topology) by prescribing that each \( K_\mathfrak{a}(P) \) is an open subring. To see what this means, it suffices (via the additive ring structure) to describe open neighborhoods of the origin. Any such neighborhood is of the form \((U_v)_{v \in M(K)} \) where \( U_v \subset K_v \) is a neighborhood of 0 for all places and \( U_v = \mathfrak{O}_v \) for all but finitely many places.

**Exercise 3:** View \( K \) as a subset of the adele ring via the diagonal embedding or canonical injection: \( \phi(\alpha) = (\alpha_v)_{v \in M(K)} \) where \( \alpha_v = \alpha \) for all places. Show that this does, indeed, take \( K \) to a subring of \( K_\mathfrak{a} \).
Lemma 2: Suppose $p$ is a prime and set $Q^{(p)}$ to be those $\alpha \in \mathbb{Q}$ where $|\alpha|_q \leq 1$ for all primes $q \neq p$. Then $Q_p = Q^{(p)} + \mathbb{Z}_p$ and $Q^{(p)} \cap \mathbb{Z}_p = \mathbb{Z}$.

Lemma 3: Set

$$Q^{(\infty)} = \mathbb{R} \times \prod_{p \text{ prime}} \mathbb{Z}_p.$$ 

Then $Q_\mathbb{A} = Q^{(\infty)} + \mathbb{Q}$ and $Q^{(\infty)} \cap \mathbb{Q} = \mathbb{Z}$, where we have identified $\mathbb{Z} \subset \mathbb{Q} \subset Q_\mathbb{A}$ via the embedding above.

Proposition 1: Via the diagonal embedding in Exercise 3 above, the field $\mathbb{Q}$ is a discrete subset of the adele ring and the quotient $Q_\mathbb{A}/\mathbb{Q}$ is compact.

Proof: First, $Q^{(\infty)}$ is clearly an open subset of $Q_\mathbb{A}$. Also, $\mathbb{Z}$ is a discrete subset of $Q^{(\infty)}$ since its projection onto the factor $\mathbb{R}$ is discrete. Certainly

$$Q^{(\infty)} = [-1/2, 1/2] \times \prod_{p \text{ prime}} \mathbb{Z}_p + \mathbb{Z}$$

so that

$$Q_\mathbb{A} = [-1/2, 1/2] \times \prod_{p \text{ prime}} \mathbb{Z}_p + \mathbb{Q}.$$ 

Since the first summand here is compact, this completes the proof.

We now turn to the case $K = \mathbb{F}_p(T)$.

Exercise 4: For every place $v \in M(K)$ set $K^{(v)}$ to be the subset of $\alpha \in K$ where $|\alpha|_w \leq 1$ for all places $w \neq v$. Prove that $K_v = K^{(v)} + \mathcal{O}_v$ and $K^{(v)} \cap \mathcal{O}_v = \mathbb{F}_p$.

Exercise 5: Prove that $K_\mathbb{A} = K_\mathbb{A}(\emptyset) + K$ and $K_\mathbb{A}(\emptyset) \cap K = \mathbb{F}_p$, where we have identified $\mathbb{F}_p \subset K \subset K_\mathbb{A}$ via the embedding above.

Together, these two exercises suffice to prove the following.

Proposition 2: Via the diagonal embedding in Exercise 3 above, $\mathbb{F}_p(T)$ is a discrete subset of its adele ring and the quotient is compact.