

Math 232 Fall 2002

Write $\vec{v} = \langle v_1, v_2, v_3 \rangle$ and $\vec{0} = \langle 0, 0, 0 \rangle$. Define

$\vec{v} \pm \vec{w} = \langle v_1, v_2, v_3 \rangle \pm \langle w_1, w_2, w_3 \rangle = \langle v_1 \pm w_1, v_2 \pm w_2, v_3 \pm w_3 \rangle$ and

$t\vec{v} = t \langle v_1, v_2, v_3 \rangle = \langle tv_1, tv_2, tv_3 \rangle$. The dot product is $\vec{v} \cdot \vec{w} = v_1w_1 + v_2w_2 + v_3w_3$.

The length is $|\vec{v}| = \sqrt{\vec{v} \cdot \vec{v}}$. With $t = 1/|\vec{v}|$, $\vec{u} = t\vec{v}$ has length 1.

Let \vec{v} and \vec{w} span two sides of a triangle. Represent the third side by $\vec{w} - \vec{v}$. Let the angle between \vec{v} and \vec{w} be given by θ . According to the Law of Cosines, it must be that

$$|\vec{w} - \vec{v}|^2 = |\vec{w}|^2 + |\vec{v}|^2 - 2|\vec{w}||\vec{v}|\cos\theta. \text{ Expand and conclude that } \vec{v} \cdot \vec{w} = |\vec{v}||\vec{w}|\cos\theta.$$

The cross product is $\vec{v} \times \vec{w} = \langle v_2w_3 - v_3w_2, v_3w_1 - v_1w_3, v_1w_2 - v_2w_1 \rangle$.

Inspection reveals that $\vec{w} \times \vec{v} = -\vec{v} \times \vec{w}$ and $\vec{v} \cdot (\vec{v} \times \vec{w}) = \vec{w} \cdot (\vec{v} \times \vec{w}) = 0$.

The length of the cross product satisfies

$$\begin{aligned} |\vec{v} \times \vec{w}|^2 &= (v_2w_3 - v_3w_2)^2 + (v_3w_1 - v_1w_3)^2 + (v_1w_2 - v_2w_1)^2 = \\ &= (v_1 + v_2 + v_3)^2 (w_1 + w_2 + w_3)^2 - (v_1w_1 + v_2w_2 + v_3w_3)^2 = |\vec{v}|^2 |\vec{w}|^2 - (\vec{v} \cdot \vec{w})^2. \end{aligned}$$

In terms of the angle θ between \vec{v} and \vec{w} this is $|\vec{v} \times \vec{w}| = |\vec{v}||\vec{w}|\sin\theta$.

Application:

Given two points P and Q let the vector \vec{p} and the vector \vec{q} go from the origin to P and Q respectively. Let $\vec{v} = \vec{q} - \vec{p}$. The parametric curve $\vec{r}(t) = \vec{p} + t\vec{v}$ traces a straight line, which passes through P when $t = 0$ and Q when $t = 1$. The midpoint of the straight line segment between P and Q is $\vec{r}(\frac{1}{2}) = \vec{p} + \frac{1}{2}\vec{v} = \frac{1}{2}(\vec{p} + \vec{q})$. If there is a mass m_P at P and a mass m_Q at Q , then the center of mass is $\frac{1}{m_P + m_Q}(m_P\vec{p} + m_Q\vec{q})$.

Application:

Let $P = (p_1, p_2, p_3)$ and $Q = (q_1, q_2, q_3)$. Connect P, Q so that $\vec{v} = \langle q_1 - p_1, q_2 - p_2, q_3 - p_3 \rangle$.

The distance between P and Q is given by

$|\vec{v}| = \sqrt{\vec{v} \cdot \vec{v}} = \sqrt{(q_1 - p_1)^2 + (q_2 - p_2)^2 + (q_3 - p_3)^2}$. All points (x, y, z) a distance $r > 0$ from P satisfy $(x - p_1)^2 + (y - p_2)^2 + (z - p_3)^2 = r^2$. This is the equation of a sphere of radius r with center at P .

Application:

Let θ be the angle between $\vec{v} = \langle v_1, v_2, v_3 \rangle$ and $\vec{w} = \langle w_1, w_2, w_3 \rangle$. It must be that

$$\theta = \cos^{-1}\left(\frac{\vec{v} \cdot \vec{w}}{|\vec{v}||\vec{w}|}\right). \text{ In particular, } \theta = \pi/2 \text{ if and only if } \vec{v} \cdot \vec{w} = 0. \text{ The vectors } \vec{v} \text{ and } \vec{w} \text{ are}$$

perpendicular if and only if $\vec{v} \cdot \vec{w} = 0$.

Application:

Let $P = (p_1, p_2, p_3)$ be some given point in a plane. Let $Q = (x, y, z)$ be an arbitrary point in the same plane. The vectors $\vec{v} = \langle x - p_1, y - p_2, z - p_3 \rangle$ are always in the plane. Let $\vec{n} \neq \vec{0}$ be a fixed vector such that $\vec{n} \cdot \vec{v} = 0$. Any such \vec{n} is a normal vector to the plane. If $\vec{n} = \langle A, B, C \rangle$, then $\vec{n} \cdot \vec{v} = 0$ is equivalent to $Ax + By + Cz = Ap_1 + Bp_2 + Cp_3 = D$.

Application:

Let θ be the angle between \vec{v} and \vec{w} . The quantity $|\vec{w}|\cos\theta$ is the size of the adjacent side, in

the direction of \vec{v} , of a triangle with hypotenuse along \vec{w} . This value is $\vec{u} \cdot \vec{w}$ where \vec{u} is a unit vector in the direction given by \vec{v} . The vector $(\vec{u} \cdot \vec{w})\vec{u}$ is the projection of \vec{w} onto \vec{v} .

Application:

Let $\vec{n} = \langle A, B, C \rangle$ be a normal vector to a plane $Ax + By + Cz = D$ with \vec{p} a vector pointing to a point (p_1, p_2, p_3) in the plane. Let \vec{u} be the unit vector in the direction given by \vec{n} . The triangle created by \vec{p} as the hypotenuse and \vec{u} along one of the short sides has $(\vec{p} \cdot \vec{u})\vec{u}$ pointing to a point in the plane. All other points in the plane must be further away from the origin. It follows that the distance from the plane to the origin is $|\vec{p} \cdot \vec{u}| = |D| / \sqrt{A^2 + B^2 + C^2}$. The distance from the plane to a point (q_1, q_2, q_3) pointed to by \vec{q} is the same formula with $\vec{p} - \vec{q}$ replacing \vec{p} so $D = A(p_1 - q_1) + B(p_2 - q_2) + C(p_3 - q_3)$.

Application:

Since $\vec{v} = t\vec{w}$ if and only if $\vec{v} \times \vec{w} = \vec{0}$, it follows that two vectors are parallel if and only if their cross product vanishes.

Application:

The area of the parallelogram spanned by \vec{v} and \vec{w} is given by $|\vec{v} \times \vec{w}|$.

Application:

The area of a triangle with vertices $P = (p_1, p_2, p_3)$, $Q = (q_1, q_2, q_3)$, and $R = (r_1, r_2, r_3)$ is $|\vec{v} \times \vec{w}|/2$, where $\vec{v} = \langle q_1 - p_1, q_2 - p_2, q_3 - p_3 \rangle$ and $\vec{w} = \langle r_1 - p_1, r_2 - p_2, r_3 - p_3 \rangle$ represents two of its sides.

Application:

The parallelepiped spanned by \vec{u} , \vec{v} and \vec{w} has volume $|\vec{u} \cdot (\vec{v} \times \vec{w})|$.

Application:

Visualize the vectors perpendicular to the line $\vec{p} + t\vec{v}$. Consider the opposite side in a right angle triangle with hypotenuse \vec{p} and adjacent side $t\vec{v}$ for some t . With θ the angle between \vec{p} and \vec{v} , the length of the opposite side is $|\vec{p}|\sin\theta$. This value is equal to $|\vec{u} \times \vec{p}|$ where \vec{u} is a unit vector in the direction given by \vec{v} . It follows that the distance from the line $\vec{p} + t\vec{v}$ to the origin is $|\vec{u} \times \vec{p}|$. The distance from the line to some arbitrary point \vec{q} is $|\vec{u} \times (\vec{p} - \vec{q})|$.

Application:

Consider the two lines $\vec{p} + t\vec{v}$ and $\vec{q} + s\vec{w}$. Visualize the triangle with hypotenuse $\vec{p} - \vec{q}$ and adjacent side the perpendicular segment between the two lines. If the lines are parallel, then the distance between the two lines equals the distance from \vec{q} to the first line. If the lines are not parallel, then the vector $\vec{v} \times \vec{w} \neq \vec{0}$ is perpendicular to both lines. Let θ be the angle between $\vec{p} - \vec{q}$ and $\vec{v} \times \vec{w}$. The length of the opposite side of the triangle is $|\vec{p} - \vec{q}|\sin\theta$. Let \vec{u} be a unit vector in the direction given by $\vec{v} \times \vec{w}$. The distance between the two lines is $|\vec{u} \times (\vec{p} - \vec{q})|$.

Application:

Suppose a line of direction \vec{v} bounces off a surface with normal \vec{n} . The reflected line's direction

\vec{w} is given by $\vec{w} = \vec{v} - 2\left(\frac{\vec{v} \cdot \vec{n}}{|\vec{n}|}\right)\frac{\vec{n}}{|\vec{n}|}$.