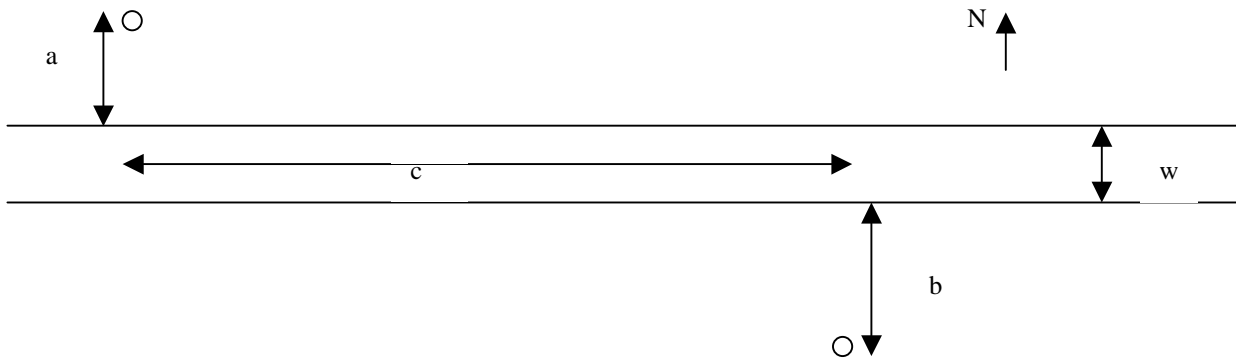


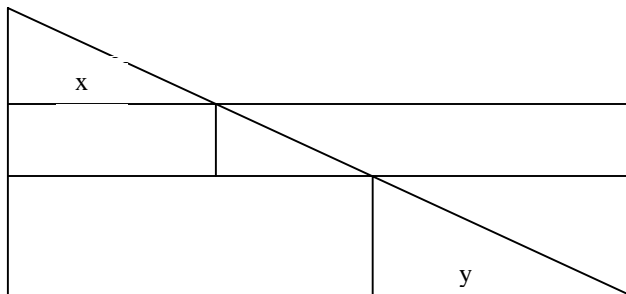
# Math 360 Sample Project Report: River Crossing



## Introduction:

The infrastructure of modern society is saturated with wiring and cabling to carry electric power and signals. Needless to say, there is often a significant cost associated with any extension of the existing network. In this report the most cost-effective configuration is determined for a buried cable connecting two locations on opposite sides of a river. It is assumed that the river crossing takes place at a straight portion of the river. The symbols denoting the various relevant distances are given in the map. The cost, per unit length, to draw cable is  $c_L > 0$  on land, and  $c_R > 0$  crossing the river.

## Problem 1:



To determine the cost when a single straight line-segment is used look at the above figure. The horizontal leg of the tiny middle triangle has length  $c - (x + y)$ . The vertical legs of the three tiny triangles are  $a, w, b$  respectively. It follows that it suffices to find  $x, y$ , and then use the Pythagorean theorem to determine the relevant lengths. To find  $x, y$ , observe that both the corresponding triangles are similar to the large triangle with legs of length  $a + w + b$  and  $c$ . It follows that

$$\frac{x}{a} = \frac{c}{a + w + b},$$

and

$$\frac{y}{b} = \frac{c}{a + w + b}.$$

After factoring out  $a, b, w$  from the square root, the cost  $C_1$  of connecting the two locations using a single straight line-segment is given by

$$C_1 = (c_L(a + b) + c_R w) \sqrt{1 + \left(\frac{c}{a + w + b}\right)^2}.$$

**Problem 2:**

The obvious technical complications due to the presence of water in the river supports the assumption that  $c_R > c_L$ . Let the number  $g$  correspond to the east-west difference between the entry and the exit points at the river. Assume the crossing is from north to south, so  $g$  is positive if the exit point is east of the entry point. The cost  $C_2$  of connecting the two locations, when  $g = c$ , is computed with the help of the Pythagorean theorem. After factoring out  $w$  the result is given by

$$C_2 = c_L(a + b) + c_R w \sqrt{1 + \left(\frac{c}{w}\right)^2}.$$

**Problem 3:**

Since  $C_3(x)$  is a continuous function and  $x \in [0, c]$ , there is a minimum. The values at the endpoints are given by

$$\begin{aligned} C_3(0) &= c_L(a + \sqrt{b^2 + c^2}) + c_R w \\ C_3(c) &= c_L(\sqrt{a^2 + c^2} + b) + c_R w \end{aligned}$$

It is immediate that  $a = b \Rightarrow C_3(0) = C_3(c)$ . Since there are no negative numbers involved it is useful to compare the squares of  $a + \sqrt{b^2 + c^2}, \sqrt{a^2 + c^2} + b$ , i.e.,

$$a^2 + 2a\sqrt{b^2 + c^2} + b^2 + c^2, a^2 + c^2 + 2b\sqrt{a^2 + c^2} + b^2,$$

or simply

$$a\sqrt{b^2 + c^2}, b\sqrt{a^2 + c^2}.$$

Square one more time and compare  $a^2(b^2 + c^2), b^2(a^2 + c^2)$ , which simplifies to  $a^2c^2, b^2c^2$ . It is now clear that

$$\begin{aligned} a > b &\Rightarrow C_3(0) > C_3(c) \\ a < b &\Rightarrow C_3(0) < C_3(c) \end{aligned}$$

The derivative of  $C_3(x)$  is given by

$$C_3'(x) = c_L \left( \frac{x}{\sqrt{a^2 + x^2}} - \frac{(c - x)}{\sqrt{b^2 + (c - x)^2}} \right).$$

The derivative is equal to zero if

$$\frac{x}{\sqrt{a^2 + x^2}} = \frac{c - x}{\sqrt{b^2 + (c - x)^2}},$$

or after squaring

$$\frac{x^2}{a^2 + x^2} = \frac{(c - x)^2}{b^2 + (c - x)^2}.$$

Algebra and simplification leads to the equation

$$\frac{a^2}{x^2} = \frac{b^2}{(c-x)^2}.$$

More algebra and applying the square root function yields

$$\frac{x}{a} = \frac{c-x}{b}.$$

It is now immediate that the derivative is zero only at the unique

$$\bar{x} = \frac{ac}{a+b}.$$

Since  $C_3'(0) < 0$  and  $C_3'(c) > 0$  it follows that the minimum is at neither of the endpoints.

Since there is a minimum and the function is differentiable, the minimum must be where the derivative is equal to zero. Create a common denominator and simplify to get the least cost  $C_3$ , when  $g = 0$ , as

$$C_3 = C_3(\bar{x}) = c_L(a+b)\sqrt{1 + \left(\frac{c}{a+b}\right)^2} + c_R w.$$

**Problem 4:**

Let  $a + b = w = c$  so that

$$C_2 = (c_L + c_R\sqrt{2})w$$

$$C_3 = (c_L\sqrt{2} + c_R)w.$$

Now choose  $c_L = 1$  and  $c_R = 2$ . Since

$$C_2/w = 1 + 2\sqrt{2} = 1 + \sqrt{2} + \sqrt{2} > 1 + \sqrt{2} + 1 = \sqrt{2} + 2 = C_3/w,$$

It follows that  $C_3 < C_2$  is possible.

Next, choose  $a + b = 1$ ,  $w = c = 100$ . This time

$$C_2 = c_L + 100c_R\sqrt{1 + \left(\frac{c}{100}\right)^2} = c_L + 100c_R\sqrt{2}.$$

$$C_3 = c_L\sqrt{1 + c^2} + 100c_R > 100(c_L + c_R)$$

Now choose  $c_L = 1$ ,  $c_R = \sqrt{2}$  so that

$$C_2 = 1 + 200 = 201$$

$$C_3 > 100(1 + \sqrt{2}) > 240.$$

It follows that  $C_3 > C_2$  is also possible.

**Problem 5:**

Let  $g \in [0, c]$  be a given fixed number. The cost  $C_5(x)$ , entering the river a distance  $x$  from the west, is by the Pythagorean theorem given by

$$C_5(x) = c_L(\sqrt{x^2 + a^2} + \sqrt{b^2 + (c-g-x)^2}) + c_R\sqrt{w^2 + g^2}.$$

The costs at the endpoints are given by

$$C_5(0) = c_L(a + \sqrt{b^2 + (c-g)^2}) + c_R\sqrt{w^2 + g^2},$$

and

$$C_5(c-g) = c_L(\sqrt{a^2 + (c-g)^2} + b) + c_R\sqrt{w^2 + g^2}.$$

The derivative is given by

$$C_5'(x) = c_L \left( \frac{x}{\sqrt{x^2 + a^2}} - \frac{c - g - x}{\sqrt{b^2 + (c - g - x)^2}} \right).$$

Note that if  $g < c$ , then  $C_5'(0) < 0$  and  $C_5'(c - g) > 0$ .

The derivative is zero if

$$\sqrt{1 + \frac{a^2}{x^2}} = \sqrt{\frac{b^2}{(c - g - x)^2} + 1},$$

or  $b^2 x^2 = a^2 (c - g - x)^2$ . It follows that  $bx = a(c - g - x)$ , and hence

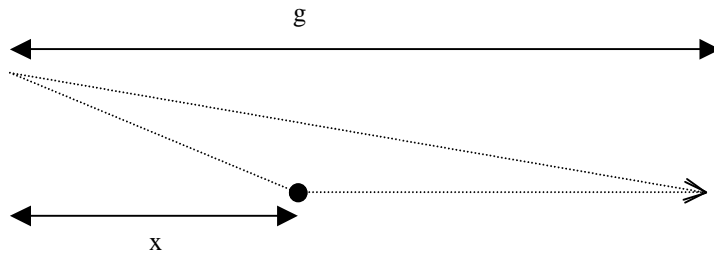
$$x = \frac{a}{a + b}(c - g).$$

Hence, for a fixed  $g$  the minimal cost is given by

$$C_{\min}(g) = C_5 \left( \frac{a}{a + b}(c - g) \right) = c_L \sqrt{(a + b)^2 + (c - g)^2} + c_R \sqrt{w^2 + g^2}.$$

**Problem 6:**

Assume  $g \in [0, c]$  is a given fixed number. Compare the costs of the two dashed paths in the picture.



The cost of the broken path is given by

$$C_6(x) = c_R \sqrt{x^2 + w^2} + c_L (g - x).$$

Let the domain be given by  $x \in [0, g]$ . The function is continuous, so to each  $g \in [0, c]$  there is a minimizing  $\bar{x} \in [0, g]$ . The minimum is either at one of the endpoints of where the derivative is zero. The derivatives are given by

$$C_6'(x) = \frac{c_R x}{\sqrt{x^2 + w^2}} - c_L, C_6''(x) = \frac{c_R w^2}{(x^2 + w^2)^{3/2}} > 0.$$

Since  $C_6'(0) = -c_L < 0$ , it follows that the minimum is never at  $x = 0$ . The derivative is zero at  $x$  only if

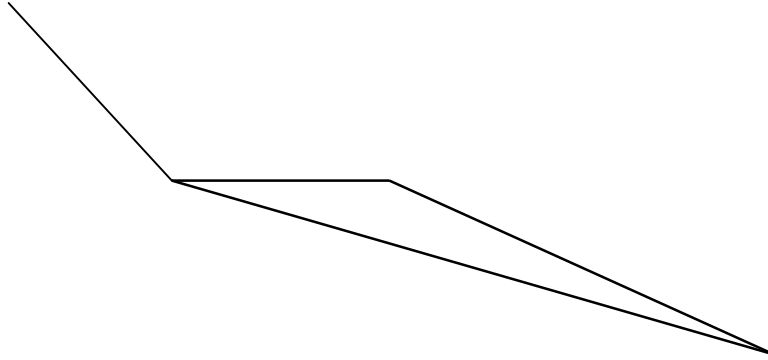
$$\frac{c_R^2 x^2}{x^2 + w^2} = c_L^2.$$

If it is assumed that  $c_R > c_L$ , then it follows that

$$x = \frac{w}{\sqrt{(c_R/c_L)^2 - 1}}.$$

It is not always the case that  $x \in [0, g]$  so the minimum is attained at

$$\bar{x} = \min \left\{ \frac{w}{\sqrt{(c_R/c_L)^2 - 1}}, g \right\}.$$



Look at the triangle formed by the path followed along the riverbed and then to the final destination, as well as the straight path from the exit point at the river to the final destination. Since all segments are on land the shortest path is the least expensive and hence the broken path is never the most cost-effective. The  $g$  corresponding to the most cost-effective path connecting the two locations must satisfy

$$g \in \left[ 0, \frac{w}{\sqrt{(c_R/c_L)^2 - 1}} \right].$$

**Problem 7:**

The derivative of  $C_{\min}(g)$  is given by

$$C'_{\min}(g) = -c_L \frac{c-g}{\sqrt{(a+b)^2 + (c-g)^2}} + c_R \frac{g}{\sqrt{w^2 + g^2}}.$$

The derivative is zero if the following holds

$$f(g) = \frac{\frac{c-g}{\sqrt{(a+b)^2 + (c-g)^2}}}{\frac{g}{\sqrt{w^2 + g^2}}} = \frac{c_R}{c_L}.$$

More algebra leads to

$$f^2(g) = \frac{1 + \frac{w^2}{g^2}}{1 + \frac{(a+b)^2}{(c-g)^2}} = \frac{c_R^2}{c_L^2}.$$

Observe that this equation can be expressed as a fourth order polynomial equation. Now it suffices to look at the numerator produced by the quotient rule to see that  $(f^2)'(g) < 0$ . It follows from the chain rule that  $2f(g)f'(g) < 0$ , and since  $f(g) > 0$  it must be that

$f'(g) < 0$ . Since  $\lim_{g \rightarrow 0^+} f(g) = +\infty$  and  $\lim_{g \rightarrow c^-} f(g) = 0$ , there is therefore a unique  $\bar{g} \in (0, c)$  such that

$$f(\bar{g}) = \frac{c_R}{c_L}.$$

It is clear that  $f^2(g)$  increases as  $c$  increases. It follows that  $\bar{g}$  increases as  $c$  increases. Specifically,

$$f^2(g) \rightarrow 1 + \frac{w^2}{g^2}$$

as  $c \rightarrow +\infty$ , and the largest  $\hat{g}$  is the solution of

$$1 + \frac{w^2}{\hat{g}^2} = \frac{c_R^2}{c_L^2},$$

i.e.,

$$\hat{g} = \frac{w}{\sqrt{(c_R/c_L)^2 - 1}}.$$

For each  $c < +\infty$  it must be that

$$\bar{g} < \frac{w}{\sqrt{(c_R/c_L)^2 - 1}}.$$

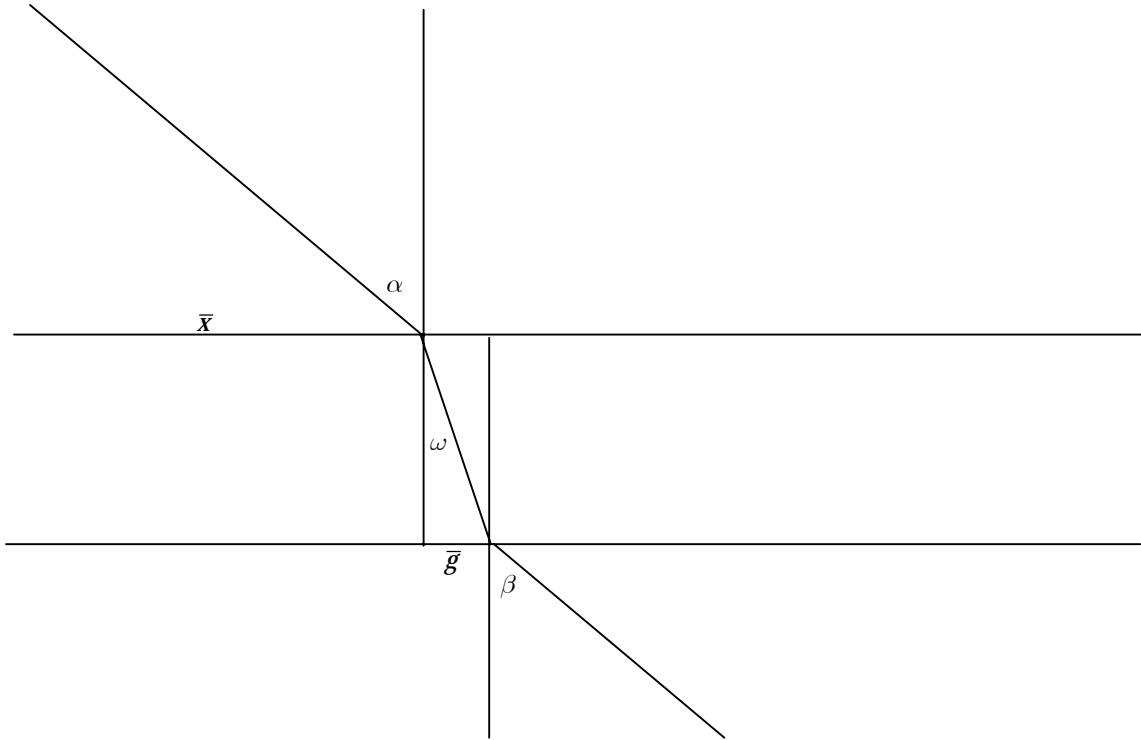
Finally, the river should be entered a distance

$$\bar{x} = \frac{a}{a+b}(c - \bar{g})$$

from the west. The cost is given by

$$C_{\min}(\bar{g}) = c_L \sqrt{(a+b)^2 + (c - \bar{g})^2} + c_R \sqrt{w^2 + \bar{g}^2}.$$

**Problem 8:**



Using trigonometry it is immediate that

$$\sin \omega = \frac{\bar{g}}{\sqrt{w^2 + \bar{g}^2}},$$

and

$$\sin \alpha = \frac{c - \bar{g}}{\sqrt{(a + b)^2 + (c - \bar{g})^2}}.$$

Then since

$$f(\bar{g}) = \frac{c_R}{c_L},$$

it follows that

$$\frac{\sin \alpha}{\sin \omega} = \frac{c_R}{c_L}.$$

Similarly, with the one leg of the triangle of length

$$c - g - \frac{a(c - g)}{a + b} = \frac{b(c - g)}{a + b},$$

it must be that  $\sin \beta = \sin \alpha$ .

The rule to follow is to mark the distance

$$\bar{x} = \frac{a(c - g)}{a + b}$$

at the river entry, and then shift the line connecting the first location to the mark keeping it parallel until it passes through the second location. The intersection of the shifted line and the river at the side of the second location gives the river exit mark.

**Conclusion:**

As the analysis shows, it suffices to solve a fourth order polynomial equation to determine the most cost-effective configuration of the cable. It is a classical mathematical fact that fourth order polynomial equations have explicit formulas for their roots. These formulas are quite cumbersome and it is better to use software, such as MATHEMATICA, to deal with these symbolic expressions. In the case of the fourth order equation in this project, MATHEMATICA generates the four possible roots as expressions that would not readily fit on a single page. For practical purposes it is probably better to calculate numerical approximations to the roots rather than dealing with these lengthy explicit expressions.