1. Dual: $\min 12\lambda_1 + 6\lambda_2 + 8\lambda_3$ when $\begin{cases} 8\lambda_1 + 2\lambda_2 \geq 1 \\ 2\lambda_1 + \lambda_2 - 4\lambda_3 \geq -2 \\ \lambda_1 + 2\lambda_3 \geq 3 \\ \lambda_1, \lambda_2, \lambda_3 \geq 0 \end{cases}$

Initial tableau:

8	2	1	1	0	0	12
2	1	0	0	1	0	6
0	−4	2	0	0	1	8
1	−2	3	0	0	0	0

.

Second tableau:

8	4	0	1	0	−1/2	8
2	1	0	0	1	0	6
0	−2	1	0	0	1/2	4
1	4	0	0	0	−3/2	−12

.

Final tableau:

2	1	0	1/4	0	−1/8	2
0	0	0	−1/4	1	1/8	4
4	0	1	1/2	0	1/4	8
−7	0	0	−1	0	−1	−20

.

The primal solution is given by $\hat{x}_1 = 0, \hat{x}_2 = 2, \hat{x}_3 = 8$, the maximum value is 20.

The dual solution is given by $\hat{\lambda}_1 = 1, \hat{\lambda}_2 = 0, \hat{\lambda}_3 = 1$, the minimum value is 20.

2. Using two bottom rows the initial tableau associated with the first phase is given by:

−1	0	1	−1	0	1	6
0	1	1	0	1	0	12
0	0	0	0	0	−1	0
−1	0	1	−1	0	0	6

Using three bottom rows the final tableaus of both the first and the second phase are given by:

−1	0	1	−1	0	1	6
1	1	0	1	1	−1	6
0	0	0	0	0	−1	0
1	2	0	0	0		0
−1	0	0	−2	0		−12

.

The feasible point $x_1 = 0, x_2 = 6, x_3 = 6$ given by the first phase is also a global maximum.

3. The slack $\hat{s}_2 = 1 > 0$, so $\hat{\lambda}_2 = 0$. From $\hat{x}_1 > 0, \hat{x}_3 > 0$ conclude that $\hat{\mu}_1 = 0, \hat{\mu}_3 = 0$.

It follows that $\begin{cases} \hat{\lambda}_1 + \hat{\lambda}_3 = 8 \\ \hat{\lambda}_3 = 7 \end{cases}$ and hence $\hat{\lambda}_1 = 1$.

4. $\hat{\lambda}^T = \begin{bmatrix} 1 & 1 & 0 \end{bmatrix}$, $c^T - \hat{\lambda}^T A = c^T - \begin{bmatrix} 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 1 & 1 \\ 3 & 2 \end{bmatrix} = c^T - \begin{bmatrix} 2 & 3 \end{bmatrix} = \begin{bmatrix} 0 & 0 \end{bmatrix}$ so $c^T = \begin{bmatrix} 2 & 3 \end{bmatrix}$.

The basic variables are given by x_2, x_1, y_3 so $c_B^T = \begin{bmatrix} 3 & 2 & 0 \end{bmatrix}$. Note that $\hat{\lambda}^T = c_B^T B^{-1}$,

$B = \begin{bmatrix} 2 & 1 & 0 \\ 1 & 1 & 0 \\ 2 & 3 & 1 \end{bmatrix}$ and $c_B^T = \begin{bmatrix} 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} 2 & 1 & 0 \\ 1 & 1 & 0 \\ 2 & 3 & 1 \end{bmatrix}$.

5. (a) The only part of the tableau that changes is the

$(c + \Delta c)^T - c_B^T B^{-1} A = c^T - c_B^T B^{-1} A + \Delta c^T = \begin{bmatrix} -\frac{1}{2} & 0 \end{bmatrix} + \begin{bmatrix} \Delta c_1 & 0 \end{bmatrix} \le \begin{bmatrix} 0 & 0 \end{bmatrix}$ so the increase allowed is at most $\frac{1}{2}$.

(b) The basic variables are given by x_2, y_2 so $c_B^T = \begin{bmatrix} 2 & 0 \end{bmatrix}$. This time

$(c^T + \Delta c^T) - (c_B^T + \Delta c_B^T) B^{-1} A = c^T - c_B^T B^{-1} A + \Delta c^T - \Delta c_B^T B^{-1} A =$

$\begin{bmatrix} -\frac{1}{2} & 0 \end{bmatrix} + \begin{bmatrix} 0 & \Delta c_2 \end{bmatrix} - \begin{bmatrix} \Delta c_2 & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{2} & 1 \\ \frac{7}{2} & 0 \end{bmatrix} = \begin{bmatrix} -\frac{1}{2} - \frac{1}{2}\Delta c_2 & 0 \end{bmatrix} \le \begin{bmatrix} 0 & 0 \end{bmatrix}$

It is also necessary to check

$-(c_B + \Delta c_B)^T B^{-1} = \begin{bmatrix} -\frac{1}{2} & 0 \end{bmatrix} - \begin{bmatrix} \Delta c_2 & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{6} & 0 \\ -\frac{1}{6} & 1 \end{bmatrix} = \begin{bmatrix} -\frac{1}{2} - \frac{1}{6}\Delta c_2 & 0 \end{bmatrix} \le \begin{bmatrix} 0 & 0 \end{bmatrix}$.

The decrease allowed is at most 1.

6. (a) $B^{-1}(b + \Delta b) = B^{-1}b + B^{-1}\Delta b = \begin{bmatrix} 90/7 \\ 75/7 \\ 95/7 \end{bmatrix} + B^{-1}\Delta b \ge 0$ leads to $\frac{90}{7} + \frac{3}{7}\Delta b_1 \ge 0$,

$\frac{75}{7} - \frac{3}{7}\Delta b_1 \ge 0$, $\frac{95}{7} - \frac{8}{21}\Delta b_1 \ge 0$, so $-30 \le \Delta b_1 \le 25$

(b) $\hat{\lambda}^T (b + \Delta b) = \hat{\lambda}^T b + \hat{\lambda}^T \Delta b = \frac{4575}{7} + \begin{bmatrix} \frac{65}{7} & \frac{15}{7} & 0 \end{bmatrix} \begin{bmatrix} 25 \\ 0 \\ 0 \end{bmatrix} = \frac{6200}{7}$

7. The dual problem is given by

$$\min\ 2\lambda_1 + 5\lambda_2 + 6\lambda_3 \text{ when } \begin{cases} 2\lambda_1 + \lambda_2 + \lambda_3 \geq 3 \\ \lambda_1 + 2\lambda_2 + 2\lambda_3 \geq 1 \\ \lambda_1 + 3\lambda_2 + \lambda_3 \geq 3 \\ \lambda_1, \lambda_2, \lambda_3 \geq 0 \end{cases}.$$

Suppose the fourth product earns profit p per item. The primal function changes to $3x_1 + x_2 + 3x_3 + px_4$. The 'new' dual problem now includes the constraint $\lambda_1 + 2\lambda_2 + 3\lambda_3 \geq p$. If the 'old' solution is still optimal then $\frac{6}{5} + 2 \cdot \frac{3}{5} + 3 \cdot 0 \geq p$.
It follows that the profit per item must exceed 2.4 before the fourth product is considered.

8. (a) The payoff matrix is given by $\begin{bmatrix} 0 & 2 & -3 & 0 \\ -2 & 0 & 0 & 3 \\ 3 & 0 & 0 & -4 \\ 0 & -3 & 4 & 0 \end{bmatrix}$

(b) $\max\ x_1 + x_2 + x_3 + x_4$ when $\begin{cases} 5x_1 + 7x_2 + 2x_3 + 5x_4 \leq 1 \\ 3x_1 + 5x_2 + 5x_3 + 8x_4 \leq 1 \\ 8x_1 + 5x_2 + 5x_3 + x_4 \leq 1 \\ 5x_1 + 2x_2 + 9x_3 + 5x_4 \leq 1 \\ x_i \geq 0,\ i \in \{1,2,3,4\} \end{cases}$

(c) Note that the value of the game is given by $v = 5$, so the probabilities are
$p_1 = 0,\ p_2 = \frac{4}{7},\ p_3 = \frac{3}{7},\ p_4 = 0$.
"Never show and say the same number" and "Show 1 and say two 57.14% of the time".

(d) $\frac{1}{4}\left(2 \cdot \frac{4}{7} - 3 \cdot \frac{3}{7}\right) + \frac{1}{4}\left(0 \cdot \frac{4}{7} + 0 \cdot \frac{3}{7}\right) + \frac{1}{4}\left(0 \cdot \frac{4}{7} + 0 \cdot \frac{3}{7}\right) + \frac{1}{4}\left(-3 \cdot \frac{4}{7} + 4 \cdot \frac{3}{7}\right) = -\frac{1}{28}$

9. Use the Kuhn-Tucker conditions. Note that $P = 1$ and
$f_1(x_1, x_2, x_3, x_4) = x_1^2 + x_2^2 + x_3^2 + x_4^2 - 1$.

(1) $\hat{\lambda} \geq 0$

(2) $\hat{\lambda}\left(\hat{x}_1^2 + \hat{x}_2^2 + \hat{x}_3^2 + \hat{x}_4^2 - 1\right) = 0$

(3) $\begin{bmatrix} 2 \\ 2 \\ 2 \\ 2\hat{x}_4 \end{bmatrix} = \lambda \begin{bmatrix} 2\hat{x}_1 \\ 2\hat{x}_2 \\ 2\hat{x}_3 \\ 2\hat{x}_4 \end{bmatrix}$

$\hat{\lambda} = 0$ is not allowed by (3). From (3) it follows that $\hat{x}_1 = \hat{x}_2 = \hat{x}_3$. If $\hat{x}_4 \neq 0$, then $\hat{\lambda} = 1$ and $\hat{x}_1 = \hat{x}_2 = \hat{x}_3 = 1$, which violates (2). The only remaining possibility is $\hat{x}_4 = 0$ and $\hat{x}_1 = \hat{x}_2 = \hat{x}_3 = \pm\frac{1}{\sqrt{3}}$. From (3) it is seen that $\hat{\lambda}$ has the same sign as $\hat{x}_1 = \hat{x}_2 = \hat{x}_3$. Hence (1) implies that $\hat{x}_1 = \hat{x}_2 = \hat{x}_3 = \frac{1}{\sqrt{3}}$ and $\hat{x}_4 = 0$, $\hat{\lambda} = \sqrt{3}$ is the only possible solution. The active constraint is regular. Since the global maximum is known to exist, the given solution is the global maximum.