

# INTRODUCTION TO OPTIMIZATION

## 1 The general one-dimensional case.

### 1.1 Notation.

The collection of all *real* numbers is denoted by  $\mathbb{R}$ . The expression  $x \in \mathbb{R}$  asserts that the symbol  $x$  represents a specific, but arbitrary, real number. The expression  $x_1, x_2 \in \mathbb{R}$  is short for the more cumbersome  $x_1 \in \mathbb{R}$  *and*  $x_2 \in \mathbb{R}$ . A real valued function of one single real variable is written as  $f: \mathbb{R} \rightarrow \mathbb{R}$ . Such a function is characterized by the fact that to each  $x \in \mathbb{R}$  there is a *unique*  $f(x) \in \mathbb{R}$ .

### 1.2 Terminology.

If  $\hat{x} \in \mathbb{R}$  has the property that  $f(\hat{x}) \geq f(x)$  for all  $x \in \mathbb{R}$ , then  $\hat{x}$  is a *maximizer* of the function  $f: \mathbb{R} \rightarrow \mathbb{R}$ .

#### Exercise

Define the term *minimizer* of a function  $f: \mathbb{R} \rightarrow \mathbb{R}$ .

A number that is either a maximizer or a minimizer of a function  $f: \mathbb{R} \rightarrow \mathbb{R}$  is called an *extremizer* of the function  $f: \mathbb{R} \rightarrow \mathbb{R}$ .

#### Exercise

Find the simplest possible function  $f: \mathbb{R} \rightarrow \mathbb{R}$  with no extremizers.

#### Exercise

Show that if there is a real number that is both a maximizer and a minimizer of  $f: \mathbb{R} \rightarrow \mathbb{R}$ , then  $f: \mathbb{R} \rightarrow \mathbb{R}$  must be a constant function.

If there is a number  $r \in \mathbb{R}$  such that  $f(x) \leq r$  for all  $x \in \mathbb{R}$ , then the function  $f: \mathbb{R} \rightarrow \mathbb{R}$  is *bounded from above*.

#### Exercise

Define what it means for a function  $f: \mathbb{R} \rightarrow \mathbb{R}$  to be *bounded from below*.

#### Exercise

Find a function  $f: \mathbb{R} \rightarrow \mathbb{R}$  of the form

$$f(x) = \begin{cases} ax + b & x < 0 \\ cx + d & 0 \leq x \end{cases}$$

that is bounded from below but with no minimizers.

A function that is both bounded from above and from below is *bounded*.

**Exercise**

Find a bounded function  $f : \mathbb{R} \rightarrow \mathbb{R}$  of the form

$$f(x) = \begin{cases} \vdots & \\ a_{-1}x + b_{-1} & -2 < x \leq -1 \\ a_0x + b_0 & -1 < x < 1 \\ a_1x + b_1 & 1 \leq x < 2 \\ \vdots & \end{cases},$$

with no extremizers.

**1.3 Continuous functions.**

A function  $f : \mathbb{R} \rightarrow \mathbb{R}$  is *continuous at a number*  $\bar{x} \in \mathbb{R}$  provided the limit

$$\lim_{x \rightarrow \bar{x}} f(x)$$

exists and is equal to  $f(\bar{x})$ . The function is *continuous* if it is continuous at each number.

**Exercise**

Find a continuous function  $f : \mathbb{R} \rightarrow \mathbb{R}$  that is bounded from below but with no minimizers.

**Exercise**

Find a bounded continuous function  $f : \mathbb{R} \rightarrow \mathbb{R}$  with no extremizers.

**Exercise**

Give an example of a continuous function  $f : \mathbb{R} \rightarrow \mathbb{R}$  with exactly two minimizers and no maximizers.

**1.4 Differentiable functions.**

Recall that a function  $f : \mathbb{R} \rightarrow \mathbb{R}$  is *differentiable at a number*  $\bar{x} \in \mathbb{R}$  provided the limit

$$\lim_{x \rightarrow \bar{x}} \frac{f(x) - f(\bar{x})}{x - \bar{x}}$$

exists. The function is *differentiable* if it is differentiable at each number. If  $f : \mathbb{R} \rightarrow \mathbb{R}$  is differentiable at  $\bar{x} \in \mathbb{R}$ , then it must be that

$$\lim_{x \rightarrow \bar{x}} f(x) = f(\bar{x}),$$

i.e.,  $f : \mathbb{R} \rightarrow \mathbb{R}$  is continuous at  $\bar{x} \in \mathbb{R}$ . It follows that if  $f : \mathbb{R} \rightarrow \mathbb{R}$  is differentiable, then  $f$  is also continuous.

If  $\bar{x}$  is a minimizer, then the numerator in the derivative limit is never negative. The right hand limit has a positive denominator and the left hand limit has a negative denominator. In order for

the limit to exist it must be that the right hand limit and the left hand limit exist and have the same value. Since the right hand limit is greater than or equal to zero and the left hand limit is less than or equal to zero, the only possibility is if the limit is zero. In short, the value of the derivative at a minimizer is necessarily zero.

### Exercise

Show that if a function  $f: \mathbb{R} \rightarrow \mathbb{R}$  is differentiable at a maximizer  $\hat{x} \in \mathbb{R}$ , then  $f'(\hat{x}) = 0$ .

### Exercise

Find the simplest example of a differentiable function  $f: \mathbb{R} \rightarrow \mathbb{R}$  such that  $f'(\bar{x}) = 0$  at some  $\bar{x} \in \mathbb{R}$  and yet there are no extremizers.

### Exercise

Give an example of a continuous function  $f: \mathbb{R} \rightarrow \mathbb{R}$  that is not differentiable.

## 1.5 Optimization issues.

There are two fundamental issues in optimization:

1. Determine if there exists an extremizer.
2. Calculate the extremizer.

To see why it is necessary to distinguish between the two issues, consider the following analogue.

The equation

$$x^3 - 17x^2 + x - 17 = 0$$

is attained from

$$(x - 17)(x^2 + 1) = 0.$$

It is clear that the only possible solution is given by  $x = 17$ . Here there is no distinction between calculating the solution and showing the existence of the solution. Now consider the equation

$$x^3 - 16x^2 + x - 17 = 0.$$

There is no longer an obvious factoring. With help from the theory of Calculus, it is still possible to prove that there exists a solution. Define

$$f(x) = x^3 - 16x^2 + x - 17,$$

and observe that  $f(0) = -17 < 0$ ,  $f(17) = 17^2 > 0$ . Polynomials are continuous and hence by the Intermediate Value Theorem there is some number  $0 < \bar{x} < 17$  such that  $f(\bar{x}) = 0$ . The value of  $\bar{x}$  is yet to be determined. For differentiable functions the fact that the derivative is zero at the extremizers leads to similar equations. In this case the issue is complicated by the fact that it is not enough to have the derivative equal to zero somewhere to establish the existence of an extremizer.

## 1.6 More terminology.

If  $C \subset \mathbb{R}$  is some given subset and  $f: \mathbb{R} \rightarrow \mathbb{R}$  a function, then write  $f: C \rightarrow \mathbb{R}$  for the *restriction* of  $f$  to the subset  $C$ . A number  $\hat{x} \in \mathbb{R}$  is a *local maximizer* of  $f: \mathbb{R} \rightarrow \mathbb{R}$  if there is

some  $r > 0$  such that  $\hat{x}$  is a maximizer of  $f : (\hat{x} - r, \hat{x} + r) \rightarrow \mathbb{R}$ . Here

$$(\hat{x} - r, \hat{x} + r) = \{x \in \mathbb{R} \mid \hat{x} - r < x < \hat{x} + r\}.$$

Similarly, it is clear how to define the concepts *local minimizer* and *local extremizer*.

### Exercise

Suppose  $f : \mathbb{R} \rightarrow \mathbb{R}$  is differentiable and  $\hat{x} \in \mathbb{R}$  is a local extremizer. Show that  $f'(\hat{x}) = 0$ .

### Exercise

Give an example of a differentiable function  $f : \mathbb{R} \rightarrow \mathbb{R}$  such that  $f'(\hat{x}) = 0$  for some  $\hat{x} \in \mathbb{R}$  but with no local extremizers.

### Exercise

Give an example of a differentiable function  $f : \mathbb{R} \rightarrow \mathbb{R}$  with at least one local extremizer but with no extremizers.

A subset  $C \subset \mathbb{R}$  is *bounded* if there is some  $r > 0$  such that  $C \subset (-r, r)$ .

### Exercise

Let  $f : \mathbb{R} \rightarrow \mathbb{R}$  be given by  $f(x) = |x|$ . Show that a subset  $C \subset \mathbb{R}$  is bounded if and only if the function  $f : C \rightarrow \mathbb{R}$  is bounded from above.

A subset  $C \subset \mathbb{R}$  is *closed* if to each  $x \notin C$  there is some  $r_x > 0$  such that  $C \cap (x - r_x, x + r_x) = \emptyset$ . This definition implies that  $\mathbb{R}$  is closed.

### Exercise

Let  $a, b \in \mathbb{R}$ . Show that  $[a, +\infty)$  and  $(-\infty, b]$  are closed. Show that  $(a, +\infty)$  and  $(-\infty, b)$  are not closed.

### Exercise

Let  $a, b \in \mathbb{R}$  satisfy  $a < b$ . Show that the intersection of two closed sets is closed, and use this to show that any interval of the form  $[a, b] = \{x \in \mathbb{R} \mid a \leq x \leq b\}$  is closed.

## 1.7 Classifying extremizers.

Suppose  $f : \mathbb{R} \rightarrow \mathbb{R}$  is differentiable with differentiable derivative. Suppose  $x_1 < \dots < x_n$  are *the only* numbers where the derivative is zero. Let  $x_0 = -\infty$  and  $x_{n+1} = +\infty$ . To determine if  $x_i$ , with  $i \in \{1, \dots, n\}$ , is a local maximizer, local minimizer, or neither, it suffices to choose  $a \in (x_{i-1}, x_i)$ ,  $b \in (x_i, x_{i+1})$  and record  $f'(a)$ ,  $f'(b)$ . If  $f'(a) < 0$  and  $f'(b) > 0$ , then  $x_i$  is a local minimizer. If  $f'(a) > 0$  and  $f'(b) < 0$ , then  $x_i$  is a local maximizer. If  $f'(a)$  and  $f'(b)$  have the same sign, then  $x_i$  is not a local extremizer. If the function has a second derivative  $f''$ ,

then the following holds. If  $f''(x_i) > 0$ , then  $x_i$  is a local minimizer. If  $f''(x_i) < 0$ , then  $x_i$  is a local maximizer. If  $f''(x_i) = 0$ , then more information is needed to draw a conclusion.

## 1.8 Existence of extremizers.

It is possible to extend the notion of closed and bounded subsets to the case of more than one variable. The notion of continuity is also available for functions of several variables. One of the truly spectacular successes of the Calculus is the following result.

### Theorem

Suppose  $C$  is closed and bounded. If  $f : C \rightarrow \mathbb{R}$  is continuous, then there exists both a maximizer and a minimizer of  $f : C \rightarrow \mathbb{R}$ .

### Exercise

Give an example of a discontinuous function, defined on a subset that is neither bounded nor closed, with exactly one minimizer and one maximizer.

### Exercise

Give an example of a function  $f : [0,1] \rightarrow \mathbb{R}$  with no extremizers.

### Exercise

Give an example of a differentiable function  $f : (0,1] \rightarrow \mathbb{R}$  with no minimizers.

### Exercise

Give an example of a differentiable function  $f : (0,1] \rightarrow \mathbb{R}$  with no extremizers.

### Exercise

A function  $f : \mathbb{R} \rightarrow \mathbb{R}$  is *coercive* if  $|x| \rightarrow +\infty$  implies  $f(x) \rightarrow +\infty$ . Show that a continuous coercive function must have a minimizer.

### Exercise

It is clear that an extremizer is always a local extremizer. Prove that if a continuous function  $f : \mathbb{R} \rightarrow \mathbb{R}$  has exactly one local extremizer, then this local extremizer is in fact an extremizer.

## 1.9 Location of extremizers.

Suppose  $C \subset \mathbb{R}$  is given. If  $x \in C$  and there is some  $r_x > 0$  such that  $(x - r_x, x + r_x) \subset C$ , then  $x \in C$  is in the *interior* of  $C$ . If  $\bar{x} \in C$  is in the interior of  $C$  and  $\bar{x}$  is a local extremizer of a differentiable  $f : \mathbb{R} \rightarrow \mathbb{R}$ , then  $f'(\bar{x}) = 0$ .

### Exercise

Suppose  $a, b \in \mathbb{R}$  satisfy  $a < b$ . Show that if  $C = [a, b]$ , then  $x \in C$  is not in the interior of  $C$  only if  $x = a$  or  $x = b$ .

For differentiable functions  $f : \mathbb{R} \rightarrow \mathbb{R}$  restricted to  $C \subset \mathbb{R}$  follow the steps:

1. Establish that there exists an extremizer.
2. Determine all points in the interior of  $C$  where the derivative is zero.
3. Compare the function values at these points with the function values at all points in  $C$  not in the interior of  $C$ .
4. Identify the extremizer.