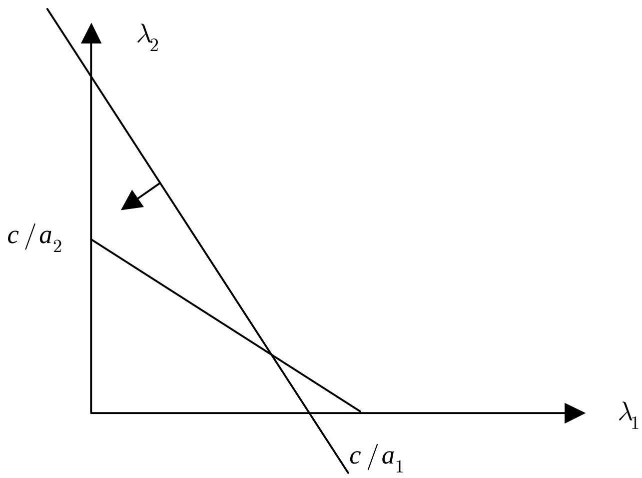
3 The linear two-dimensional case.

3.1 Two variables and one extra constraint.

Consider the dual problem

$$\min b_1\lambda_1 + b_2\lambda_2 \text{ when } \begin{cases} a_1\lambda_1 + a_2\lambda_2 \geq c \\ \lambda_1, \lambda_2 \geq 0 \end{cases}.$$

Since only two variables are involved, it is possible to draw a useful picture. Assume the data is such that the following picture is relevant



The feasible set extends without bound in the first quadrant. The feasible set has two vertices, one at $(c/a_1, 0)$ and the other at $(0, c/a_2)$. A level line of $b_1\lambda_1 + b_2\lambda_2$ is also indicated. Recall that a level line is the collection of all points (λ_1, λ_2) such that $b_1\lambda_1 + b_2\lambda_2$ takes on the same value. This value is referred to as the level. All level lines of $b_1\lambda_1 + b_2\lambda_2$ are parallel. In the picture an arrow is attached to the level line to indicate where the line 'moves' if the level decreases. As the level decreases there is eventually only one point that is both feasible and on the level line. In the picture this point is $(0, c/a_2)$. If the level is further reduced, then there are no feasible points on the level line. From this observation it is geometrically clear that the minimum is attained at the point $(0, c/a_2)$. The slope of any level line is given by $-b_1/b_2$. The slope of the boundary line of the feasible set is $-a_1/a_2$. To be consistent with the picture it must be that $-b_1/b_2 < -a_1/a_2$ or $b_2/a_2 < b_1/a_1$.

3.2 Primal problem.

Now look at the corresponding primal problem:

$$\max cx \text{ when } \begin{cases} \mathrm{a}_1 x \le b_1 \\ \mathrm{a}_2 x \le b_2 \\ x \ge 0 \end{cases}.$$

The initial tableau is given by

$$\begin{array}{ccc|c} a_1 & 1 & 0 & b_1 \\ a_2 & 0 & 1 & b_2 \\ \hline c & 0 & 0 & 0 \end{array}.$$

Use the pivoting rule to get the final tableau

$$\begin{array}{ccc|c} 0 & 1 & -\dfrac{a_1}{a_2} & b_1 - a_1\dfrac{b_2}{a_2} \\ 1 & 0 & \dfrac{1}{a_2} & \dfrac{b_2}{a_2} \\ \hline 0 & 0 & -\dfrac{c}{a_2} & -c\dfrac{b_2}{a_2} \end{array}.$$

As expected the final tableau produces $\lambda_1 = 0$ and $\lambda_2 = c / a_2$.

3.3 Two variables and any finite number of extra constraints.

Each constraint of the form $a_1x_1 + a_2x_2 \le b$ is satisfied by a collection of points contained in a closed half-plane. Both the primal and the dual problem in two variables have a feasible set that is the intersection of a finite number of closed half-planes. Since the variables are assumed non-negative, the feasible set is always a subset of the first quadrant. No optimal point is in the interior of the feasible set so at least one of the constraints must hold with equality. If a constraint is satisfied with equality at a point, then the constraint is said to be *active* at that point. Now, suppose there is an optimal point such that only one constraint is active. A level line of the objective must be parallel to the line given by the active constraint because otherwise it is possible to increase or decrease the objective. It follows that all feasible points on the line given by the active constraint are optimal. There is no line completely contained in the first quadrant so some point on the line given by the active constraint must also be on the line of a different constraint. Hence, to locate an optimal point it suffices to examine the points of the feasible set where at least two constraints are active. Points where several constraints are active are of course vertices of the feasible set.

Exercise

Express the line segment from $(0,3)$ to $(5,0)$ as a feasible set of a primal problem.

Exercise

Consider the statement ‘If there is an optimal point, then there must be an optimal point at a point where two or more constraints are active.’ Give an example that shows that the statement is false if it is not assumed that the variables are nonnegative.

Exercise

Give an example of a primal problem such that there are no extremizers.