

5 The general linear optimization problem.

5.1 Primal and dual problem.

Let $n, m \in \mathbb{N}$. Assume that $x, c \in \mathbb{R}^n$ and $\lambda, b \in \mathbb{R}^m$. Let A be an $m \times n$ real matrix. The primal problem is given by

$$\max c^T x \text{ when } \begin{cases} Ax \leq b \\ x \geq 0 \end{cases}.$$

The corresponding dual problem is given by

$$\min \lambda^T b \text{ when } \begin{cases} A^T \lambda \geq c \\ \lambda \geq 0 \end{cases}.$$

5.2 Verifiable solutions.

Whenever a ‘solution’ is given for an optimization problem it is prudent to be suspicious. In the case of linear problems there is a wonderful optimality test available. To apply the test both the primal problem and the corresponding dual problem must be solved. General-purpose programs such as Solver in Excel capably handle both problems. To verify a solution-pair one must check one equality and $2(n + m)$ inequalities. The test is based on the following result.

VERIFICATION THEOREM

If $\hat{x} \in \mathbb{R}^n$ is feasible in the primal problem and $\hat{\lambda} \in \mathbb{R}^m$ is feasible in the dual problem, then $c^T \hat{x} = \hat{\lambda}^T b$ implies that both \hat{x} and $\hat{\lambda}$ are optimal.

Proof:

Suppose $x \in \mathbb{R}^n$ is any feasible point in the dual problem, then

$$c^T x \leq (A^T \hat{\lambda})^T x = (\hat{\lambda}^T (A^T)^T) x = (\hat{\lambda}^T A) x = \hat{\lambda}^T (Ax) \leq \hat{\lambda}^T b = c^T \hat{x},$$

and hence $\hat{x} \in \mathbb{R}^n$ is optimal in the primal problem.

Suppose $\lambda \in \mathbb{R}^m$ is any feasible point in the dual problem, then

$$\lambda^T b \geq \lambda^T (A\hat{x}) = (\lambda^T A) \hat{x} = (A^T \lambda)^T \hat{x} \geq c^T \hat{x} = \hat{\lambda}^T b,$$

and hence $\hat{\lambda} \in \mathbb{R}^m$ is optimal in the dual problem. □

Exercise

Examine the proof of the Verification Theorem. Pinpoint where in the proof the fact that \hat{x} satisfies $\hat{x} \geq 0$ is used. Pinpoint where in the proof the fact that $\hat{\lambda}$ satisfies $\hat{\lambda} \geq 0$ is used.

5.3 Equal objectives.

Suppose $\mathbf{x} \in \mathbb{R}^n$ is feasible in the primal problem and $\lambda \in \mathbb{R}^m$ is feasible in the dual problem. The dual objective always exceeds the primal objective since

$$\mathbf{c}^T \mathbf{x} \leq (\mathbf{A}^T \lambda)^T \mathbf{x} = (\lambda^T (\mathbf{A}^T)^T) \mathbf{x} = (\lambda^T \mathbf{A}) \mathbf{x} = \lambda^T (\mathbf{A} \mathbf{x}) \leq \lambda^T \mathbf{b}.$$

If $\hat{\lambda} \in \mathbb{R}^m$ is optimal in the dual problem, then $\mathbf{c}^T \mathbf{x} \leq \hat{\lambda}^T \mathbf{b}$ for all feasible points $\mathbf{x} \in \mathbb{R}^n$ of the primal problem. If $\hat{\mathbf{x}} \in \mathbb{R}^n$ is optimal in the primal problem, then $\mathbf{c}^T \hat{\mathbf{x}} \leq \hat{\lambda}^T \mathbf{b}$. Observe that the Verification Theorem is not applicable if $\mathbf{c}^T \hat{\mathbf{x}} < \hat{\lambda}^T \mathbf{b}$. It is therefore very exciting to announce that $\mathbf{c}^T \hat{\mathbf{x}} < \hat{\lambda}^T \mathbf{b}$ never happens at optimal points. This fact is one of the consequences of the so-called Fundamental Theorem of linear optimization. The Fundamental Theorem is an example of a ‘deeper’ mathematical result, and its details are deferred for now.

5.4 Complementary Slackness.

COMPLEMENTARY SLACKNESS THEOREM

Assume $\hat{\mathbf{x}} \in \mathbb{R}^n$ is feasible in the primal problem and $\hat{\lambda} \in \mathbb{R}^m$ is feasible in the dual problem. Let $\hat{\mathbf{y}} = \mathbf{b} - \mathbf{A}\hat{\mathbf{x}} \in \mathbb{R}^m$ be the slack in the primal problem and $\hat{\mu} = \mathbf{A}^T \hat{\lambda} - \mathbf{c} \in \mathbb{R}^n$ the slack in the dual problem. Suppose that $\mathbf{c}^T \hat{\mathbf{x}} = \hat{\lambda}^T \mathbf{b}$ so that $\hat{\mathbf{x}}$ and $\hat{\lambda}$ are optimal, then

- (1) $\hat{y}_j > 0 \Rightarrow \hat{\lambda}_j = 0, \quad j \in \{1, \dots, m\}$
- (2) $\hat{\mu}_i > 0 \Rightarrow \hat{x}_i = 0, \quad i \in \{1, \dots, n\}$
- (3) $\hat{x}_i > 0 \Rightarrow \hat{\mu}_i = 0, \quad i \in \{1, \dots, n\}$
- (4) $\hat{\lambda}_j > 0 \Rightarrow \hat{y}_j = 0, \quad j \in \{1, \dots, m\}$

Proof:

First observe that

$$(\mathbf{A}^T \hat{\lambda} - \mathbf{c})^T \hat{\mathbf{x}} + \hat{\lambda}^T (\mathbf{b} - \mathbf{A}\hat{\mathbf{x}}) = 0,$$

and

$$(\mathbf{A}^T \hat{\lambda} - \mathbf{c})^T \hat{\mathbf{x}} + \hat{\lambda}^T (\mathbf{b} - \mathbf{A}\hat{\mathbf{x}}) = \hat{\mu}^T \hat{\mathbf{x}} + \hat{\lambda}^T \hat{\mathbf{s}},$$

so

$$\hat{\mu}^T \hat{\mathbf{x}} + \hat{\lambda}^T \hat{\mathbf{s}} = 0.$$

Every single term in both of the matrix products is non-negative and therefore equal to zero. Each term is a product of two factors. If one factor is greater than zero, then the other factor must be equal to zero. □

Example

The primal problem

$$\max \mathbf{x}_1 - 2\mathbf{x}_2 + 3\mathbf{x}_3 \quad \text{when} \quad \begin{cases} 4\mathbf{x}_1 + 5\mathbf{x}_2 + 6\mathbf{x}_3 \leq 7 \\ \mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3 \geq 0 \end{cases}$$

has dual

$$\min 7\lambda \text{ when } \begin{cases} 4\lambda \geq 1 \\ 5\lambda \geq -2 \\ 6\lambda \geq 3 \\ \lambda \geq 0 \end{cases} .$$

The solution of the dual problem is given by $\hat{\lambda} = \frac{1}{2}$. The slack is given by

$$\hat{\mu} = \begin{bmatrix} 1 \\ \frac{9}{2} \\ 0 \end{bmatrix} .$$

Using complementary slackness it follows that $\hat{x}_1 = 0$ and $\hat{x}_2 = 0$. It also follows that $\hat{y}_1 = 0$ and $4\hat{x}_1 + 5\hat{x}_2 + 6\hat{x}_3 = 7$, and hence $\hat{x}_3 = \frac{7}{6}$.

5.5 Conversion.

Suppose $\hat{x} \in \mathbb{R}^n$ is optimal with $\hat{y} \in \mathbb{R}^m$ the corresponding slack. If the augmented point is a basic *non-degenerate* solution so that

$$\begin{bmatrix} \hat{x} \\ \hat{y} \end{bmatrix}$$

is such that exactly n entries are zero and all other entries are positive, then the dual solution is found from the linear system $A^T \hat{\lambda} - \hat{\mu} = c$. This system has $n + m$ variables and n equations. By the complementary slackness m of the variables are equal to zero. The remaining n variables are given as a solution to the system.