7 The simplex algorithm by example.

7.1 A second pivoting rule.

Consider the primal problem

$$\max 2x_1 + 3x_2 \text{ when } \begin{cases} x_1 + 3x_2 \leq 9 \\ 2x_1 + x_2 \leq 8 \\ x_1, x_2 \geq 0 \end{cases}.$$

The corresponding initial tableau is given by

$$\begin{array}{cccc|c} 1 & 3 & 1 & 0 & 9 \\ 2 & 1 & 0 & 1 & 8 \\ \hline 2 & 3 & 0 & 0 & 0 \end{array}.$$

Since both x_1 and x_2 are non-basic variables, it is possible to pivot using either variable. It is reasonable to examine the current objective and choose the variable with the largest coefficient. In the example x_2 has the larger coefficient. In terms of the tableau, the second pivoting rule selects the column with the largest value in the bottom row to the left of the vertical line. This choice leads to the 'fastest' increase in the objective.

7.2 The next tableau.

Combine the two pivoting rules and pivot to get

$$\begin{array}{cccc|c} 1/3 & 1 & 1/3 & 0 & 3 \\ 5/3 & 0 & -1/3 & 1 & 5 \\ \hline 1 & 0 & -1 & 0 & -9 \end{array}.$$

The new solution is given by $x_1 = 0, x_2 = 3$. There is a positive coefficient in the 'new' objective $9 + x_1 - y_1$. There is a need for a second pivot.

7.3 The final tableau.

Apply the original pivoting rule and get the tableau

$$\begin{array}{cccc|c} 0 & 1 & 2/5 & -1/5 & 2 \\ 1 & 0 & -1/5 & 3/5 & 3 \\ \hline 0 & 0 & -4/5 & -3/5 & -12 \end{array}.$$

The 'new' objective is given by $12 - \frac{4}{5}y_1 - \frac{3}{5}y_2$ with no hope of improvement. This must be the final tableau and the proposed solution is given by the feasible $\hat{x}_1 = 3, \hat{x}_2 = 2$. To verify the solution observe that the dual problem is given by

$$\min 9\lambda_1 + 8\lambda_2 \text{ when } \begin{cases} \lambda_1 + 2\lambda_2 \geq 2 \\ 3\lambda_1 + \lambda_2 \geq 3 \\ \lambda_1, \lambda_2 \geq 0 \end{cases}.$$

The tableau suggests $\hat{\lambda}_1 = \frac{4}{5}, \hat{\lambda}_2 = \frac{3}{5}$, which is indeed feasible. Since $9\hat{\lambda}_1 + 8\hat{\lambda}_2 = 2\hat{x}_1 + 3\hat{x}_2$ the verification theorem guarantees that both solutions are optimal.