

8 The two-phase method.

8.1 The two-phase method.

The simplex algorithm assumes that the initial point is feasible in the primal problem. If b is greater than or equal to zero, then the origin is feasible. If the origin is not feasible, then it is necessary to determine some other initial point that is feasible. It is possible to introduce an auxiliary primal problem specifically designed to help in this task. The first phase of the two-phase method applies the simplex algorithm to the auxiliary problem. The second phase uses the feasible point generated in the first phase as initial point in the original problem. There is typically a need for elementary row operations to bring the tableau into the form required by the simplex algorithm.

8.2 The auxiliary problem.

Consider the primal problem

$$\max -9x_1 - 8x_2 \text{ when } \begin{cases} -x_1 - 2x_2 \leq -2 \\ -3x_1 - x_2 \leq -3 \\ x_1, x_2 \geq 0 \end{cases}$$

The feasible set does not include the origin. The augmented system has the two equations

$$\begin{cases} -x_1 - 2x_2 + y_1 = -2 \\ -3x_1 - x_2 + y_2 = -3 \end{cases}$$

Write the system in the form

$$\begin{cases} x_1 + 2x_2 - y_1 + z_1 = 2 \\ 3x_1 + x_2 - y_2 + z_2 = 3 \end{cases}$$

using the auxiliary non-negative variables z_1, z_2 . The auxiliary problem seeks to minimize $z_1 + z_2$ without violating the previous system.

8.3 Tableaux.

If the minimizer is given by $z_1 = z_2 = 0$, then the previous system yields a solution that is feasible in the original problem. The tableau associated with the problem to maximize $-z_1 - z_2$ is given by

$$\begin{array}{cccccc|c} 1 & 2 & -1 & 0 & 1 & 0 & 2 \\ 3 & 1 & 0 & -1 & 0 & 1 & 3 \\ \hline 0 & 0 & 0 & 0 & -1 & -1 & 0 \end{array}$$

Observe how there are no basic variables. This is readily fixed by adding the first and the second row to the last row. It is convenient to extend the tableau with one more row for this particular

step. The extended tableau is given by

$$\begin{array}{cccccc|c} 1 & 2 & -1 & 0 & 1 & 0 & 2 \\ 3 & 1 & 0 & -1 & 0 & 1 & 3 \\ \hline 0 & 0 & 0 & 0 & -1 & -1 & 0 \\ \hline 4 & 3 & -1 & -1 & 0 & 0 & 5 \end{array}.$$

8.4 The auxiliary problem and the simplex method.

The simplex algorithm is now applicable and the next tableau is given by

$$\begin{array}{cccccc|c} 0 & 5/3 & -1 & 1/3 & 1 & -1/3 & 1 \\ 1 & 1/3 & 0 & -1/3 & 0 & 1/3 & 1 \\ \hline 0 & 5/3 & -1 & 1/3 & 0 & -4/3 & 1 \end{array}.$$

The algorithm calls for one more iteration. The next tableau is given by

$$\begin{array}{cccccc|c} 0 & 1 & -3/5 & 1/5 & 3/5 & -1/5 & 3/5 \\ 1 & 0 & 1/5 & -2/5 & -1/5 & 2/5 & 4/5 \\ \hline 0 & 0 & 0 & 0 & -1 & -1 & 0 \end{array}.$$

Observe how the value of the objective has increased from -5 to -1 and finally 0 . The auxiliary variables are both non-basic and hence zero. They have served their role and produced the feasible point $x_1 = 4/5, x_2 = 3/5$.

8.5 The original problem and the simplex method.

It is now time to bring back the original objective and delete the auxiliary variables. The tableau is given by

$$\begin{array}{cccc|c} 0 & 1 & -3/5 & 1/5 & 3/5 \\ 1 & 0 & 1/5 & -2/5 & 4/5 \\ \hline -9 & -8 & 0 & 0 & 0 \end{array}.$$

There are again no basic variables. To fix this, add eight times the first and nine times the second row to the last row. The extended tableau is given by

$$\begin{array}{cccc|c} 0 & 1 & -3/5 & 1/5 & 3/5 \\ 1 & 0 & 1/5 & -2/5 & 4/5 \\ \hline -9 & -8 & 0 & 0 & 0 \\ \hline 0 & 0 & -3 & -2 & -12 \end{array}.$$

This is in fact a final tableau and $x_1 = 4/5, x_2 = 3/5$ is optimal.