

9 Matrix form of the simplex algorithm.

9.1 Initial tableau.

The primal problem

$$\max c^T x \text{ when } \begin{cases} Ax \leq b \\ x \geq 0 \end{cases},$$

with $x \in \mathbb{R}^n$, $b \in \mathbb{R}^m$, so that A is $m \times n$, yields an augmented system of the form

$$\begin{bmatrix} A & I \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = b.$$

To keep the notation simple, continue to write 0 for each matrix with all entries 0, and also I for each identity matrix. The order of any matrix can be deduced from its relative position in the system. For instance, the I in the augmented system must be $m \times m$. With this in mind, the initial tableau has the form

$$\begin{array}{cc|c} A & I & b \\ \hline c^T & 0 & 0 \end{array}.$$

The simplex algorithm assumes that

$$\begin{bmatrix} 0 \\ b \end{bmatrix}$$

is a feasible solution of the augmented system. Each x_i in x is a non-basic variable, and each y_j in y is a basic variable. It is of course in general not true that $b \geq 0$, but in this case the first phase of the two-phase method generates a tableau in the form consistent with the simplex algorithm.

9.2 Subsequent tableaux.

The application of elementary row operations to convert a non-basic variable to a basic variable is referred to as *pivoting*. Each elementary row operation corresponds to a matrix, and if several operations are applied in succession, then some matrix again represents the combined effect. To avoid losing track of the meaning of the various matrices use the following example.

Example

Consider the primal problem

$$\max \begin{bmatrix} 2 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \text{ when } \begin{cases} \begin{bmatrix} 3 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \leq \begin{bmatrix} 9 \\ 5 \end{bmatrix} \\ \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \geq \begin{bmatrix} 0 \\ 0 \end{bmatrix} \end{cases}.$$

Here $c^T = \begin{bmatrix} 2 & 1 \end{bmatrix}$, $b = \begin{bmatrix} 9 \\ 5 \end{bmatrix}$, and $A = \begin{bmatrix} 3 & 1 \\ 1 & 1 \end{bmatrix}$.

The vertices of the feasible set are $(0,0)$, $(3,0)$, $(2,3)$ and $(0,5)$. The initial tableau

$$\begin{array}{cc|c} A & I & b \\ \hline c^T & 0 & 0 \end{array}$$

is given by

$$\begin{array}{cccc|c} 3 & 1 & 1 & 0 & 9 \\ 1 & 1 & 0 & 1 & 5 \\ \hline 2 & 1 & 0 & 0 & 0 \end{array}$$

Here x_1, x_2 are non-basic variables, and the slack variables y_1, y_2 are basic variables. The values are given by $x_1 = 0$, $x_2 = 0$, and $y_1 = 9$, $y_2 = 5$. Four different pivots are possible. Two of the pivots turn x_1 into a basic variable. The pivot on the first row yields

$$\begin{array}{cccc|c} 1 & 1/3 & 1/3 & 0 & 3 \\ 0 & 2/3 & -1/3 & 1 & 2 \\ \hline 0 & 1/3 & -2/3 & 0 & -6 \end{array}.$$

Now x_1, y_2 are basic variables and x_2, y_1 are non-basic variables. The values are given by $x_1 = 3$, $x_2 = 0$, and $y_1 = 0$, $y_2 = 2$. Geometrically the pivot turns the attention away from the vertex $(0,0)$ and directs it to the vertex $(3,0)$. Since the identity matrix has been replaced by

$$\begin{bmatrix} 1/3 & 0 \\ -1/3 & 1 \end{bmatrix},$$

it follows that this matrix represents the combined effect of the elementary row operations. A quick check confirms that

$$\begin{bmatrix} 1/3 & 0 \\ -1/3 & 1 \end{bmatrix} \begin{bmatrix} 3 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1/3 \\ 0 & 2/3 \end{bmatrix}.$$

Instead of observing the change of the identity matrix in the initial tableau, examine how the identity matrix in the new tableau is created. It must be that

$$\begin{bmatrix} 1/3 & 0 \\ -1/3 & 1 \end{bmatrix} \begin{bmatrix} 3 & 0 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},$$

and hence the combined effect of the elementary row operations is represented by the inverse of the matrix

$$\begin{bmatrix} 3 & 0 \\ 1 & 1 \end{bmatrix}.$$

This matrix is given by the columns in the initial tableau corresponding to the new basic variables in the appropriate order. Let B denote this matrix and let N denote the matrix consisting of all the remaining columns in $[A \ I]$ in order from left to right. In the present example N is given by

$$N = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}.$$

Similarly, let

$$c_{BN}^T = [c \ 0] = [2 \ 1 \ 0 \ 0],$$

and

$$c_B^T = [2 \ 0], c_N^T = [1 \ 0].$$

Let

$$z = \begin{bmatrix} x \\ y \end{bmatrix}$$

be the *current* solution so $z^T = [3 \ 0 \ 0 \ 2]$. Let z_B be the part of z corresponding to the basic variables, and z_N the zero matrix corresponding to the non-basic variables so

$$z_B = \begin{bmatrix} 3 \\ 2 \end{bmatrix}, z_N = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

A careful examination of the pivoting process reveals that each tableau has the form

$$\begin{array}{cc|c} B^{-1}A & B^{-1} & B^{-1}b \\ \hline c^T - c_B^T B^{-1}A & -c_B^T B^{-1} & -c_B^T B^{-1}b \end{array}.$$

To check this compute

$$-c_B^T B^{-1} = -[2 \ 0] \begin{bmatrix} \frac{1}{3} & 0 \\ -\frac{1}{3} & 1 \end{bmatrix} = [-\frac{2}{3} \ 0],$$

$$-c_B^T B^{-1}b = -[\frac{2}{3} \ 0] \begin{bmatrix} 9 \\ 5 \end{bmatrix} = [-6],$$

and

$$c^T - c_B^T B^{-1}A = [2 \ 1] - [\frac{2}{3} \ 0] \begin{bmatrix} 3 & 1 \\ 1 & 1 \end{bmatrix} = [2 \ 1] - [2 \ \frac{2}{3}] = [0 \ \frac{1}{3}].$$

The current objective is given by $c^T x = c_{BN}^T z = c_B^T z_B + c_N^T z_N$. Since $Bz_B + Nz_N = b$, it follows that $z_B = B^{-1}b - B^{-1}Nz_N$, and $c^T x = c_B^T B^{-1}b + (c_N^T - c_B^T B^{-1}N)z_N = c_B^T B^{-1}b$. The bottom row has zeroes in the columns corresponding to the basic variables. The entries corresponding to the non-basic variables in the bottom row, when written in order from left to right, yield a matrix equal to $c_N^T - c_B^T B^{-1}N$. To check this, compute

$$c_N^T - c_B^T B^{-1}N = [1 \ 0] - [2 \ 0] \begin{bmatrix} \frac{1}{3} & 0 \\ -\frac{1}{3} & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} = [1 \ 0] - [\frac{2}{3} \ 0] \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} = [\frac{1}{3} \ -\frac{2}{3}].$$

The pivot on the second row yields

$$\begin{array}{cccc|c} 0 & -2 & 1 & -3 & -6 \\ 1 & 1 & 0 & 1 & 5 \\ \hline 0 & -1 & 0 & -2 & -10 \end{array}.$$

This time y_1, x_1 are basic variables and x_2, y_2 are non-basic variables. The values are given by $x_1 = 5$, $x_2 = 0$, and $y_1 = -6$, $y_2 = 0$. The point $(5, 0)$ is not a vertex of the feasible set. It does, however, correspond to a solution of the augmented system. The fact that one of the slack

variables is negative shows that one of the constraints is violated. Since $9/3 < 5/1$, the pivot rule selects the first row as opposed to the second row. The row operations are represented by

$$\begin{bmatrix} 1 & -3 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix}^{-1} = B^{-1}.$$

Observe how the order of the columns in the inverse matches the order of the basic variables in the new tableau. The corresponding columns form the identity matrix in that order. This time

$$N = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix},$$

and $c_B^T = [0 \ 2]$, $c_N^T = [1 \ 0]$. It follows that

$$c_B^T B^{-1} b = [0 \ 2] \begin{bmatrix} 1 & -3 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 9 \\ 5 \end{bmatrix} = [0 \ 2] \begin{bmatrix} 9 \\ 5 \end{bmatrix} = [10],$$

and

$$c_N^T - c_B^T B^{-1} N = [1 \ 0] - [0 \ 2] \begin{bmatrix} 1 & -3 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} = [1 \ 0] - [0 \ 2] \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} = [-1 \ -2].$$

Exercise

Compute the next tableau by honoring the pivot rule as it applies to the second column of

$$\begin{array}{cccc|c} 3 & 1 & 1 & 0 & 9 \\ 1 & 1 & 0 & 1 & 5 \\ \hline 2 & 1 & 0 & 0 & 0 \end{array}$$

Extract all the information as in the preceding example. Next, compare this with the tableau and the conclusions reached when the pivot rule is violated.

9.3 Optimality condition.

Given a tableau with $\{k_1, \dots, k_m\}$, such that $1 \leq k_j \leq n + m$, the indices of the basic variables in the order corresponding to the identity matrix. Let B be the $m \times m$ matrix created from $[A \ I]$ by selecting the columns $\{k_1, \dots, k_m\}$ in this order. Let N be the matrix formed by the remaining columns in order from left to right. Let $c_{BN}^T = [c \ 0]$ be of order $1 \times (n + m)$. Define c_B^T by selecting from c_{BN}^T the entries corresponding to $\{k_1, \dots, k_m\}$ preserving this order. Let c_N^T be the matrix formed by the remaining entries in order from left to right. Let $z = \begin{bmatrix} x \\ y \end{bmatrix}$ and define z_N in the same manner. Again, since $Bz_B + Nz_N = b$, it follows that $z_B = B^{-1}b - B^{-1}Nz_N$, and $c^T x = c_B^T B^{-1}b + (c_N^T - c_B^T B^{-1}N)z_N = c_B^T B^{-1}b$. The current solution has $z_N = 0$, and each competing solution has $z_N \geq 0$. It follows that if $c_N^T - c_B^T B^{-1}N \leq 0$, then the current solution is optimal. In the tableau this corresponds to no positive values in the row below the horizontal line.

9.4 Duality.

The optimality condition is equivalent to $c_B^T B^{-1} N \geq c_N^T$. A feasible point λ in the dual problem satisfies $A^T \lambda \geq c$ and $\lambda \geq 0$. This is equivalent to $\lambda^T A \geq c^T$ and $\lambda^T \geq 0$. Using the augmented matrix this is equivalent to $\lambda^T [A \ I] \geq [c^T \ 0]$. A reordering leads to $\lambda^T [B \ N] \geq [c_B^T \ c_N^T]$. This last inequality is equivalent to $[\lambda^T B \ \lambda^T N] \geq [c_B^T \ c_N^T]$. When this is compared with $c_B^T B^{-1} N \geq c_N^T$, the choice $\lambda^T = c_B^T B^{-1}$ stands out. It is only a matter of backtracking through the equivalent inequalities to see that $\lambda^T = c_B^T B^{-1}$ is feasible in the dual problem. The value of the primal function at the basic solution $z_B = B^{-1}b$, $z_N = 0$ is given by $c_B^T B^{-1}b$. The value of the dual function at $\lambda^T = c_B^T B^{-1}$ is given by $c_B^T B^{-1}b$. The two values are equal and by the Verification Theorem, $\lambda^T = c_B^T B^{-1}$ is a global minimum of the dual problem.

The initial tableau is given by

$$\begin{array}{cc|c} A & I & b \\ \hline c^T & 0 & 0 \end{array}.$$

Each subsequent tableau has the form

$$\begin{array}{cc|c} B^{-1}A & B^{-1} & B^{-1}b \\ \hline c^T - c_B^T B^{-1}A & -c_B^T B^{-1} & -c_B^T B^{-1}b \end{array}.$$

This is the same as

$$\begin{array}{cc|c} B^{-1}A & B^{-1} & z_B \\ \hline -\mu^T & -\lambda^T & -\lambda^T b \end{array},$$

where $\mu = A^T \lambda - c$. Also note that $\lambda^T b = c_B^T z_B = c^T x$. It is only if $z_B \geq 0$, $\lambda \geq 0$, and $\mu \geq 0$ that λ an optimal solution of the dual problem.