10 Sensitivity.

10.1 Issues.

The solution of a linear programming problem yields an optimal feasible point. It is often necessary to be concerned about how changes in the data A, b, c affect the solution. It is particularly troubling if tiny changes in the data radically alter the optimal solution. The search for these so-called *sensitive* values is therefore a worthwhile pursuit. Simple examples of sources of change in the data are market fluctuations and measurement errors. In contrast to these uncontrolled changes are controlled changes in the data. Examples include changes in the available resources, price alterations, and efficiency modifications. Sensitivity analysis is carried out to get quantitative information about the effect of changes in the data.

10.2 Changes in the resources.

Let the original b be replaced by $\overline{b}$. Use the notation $\Delta b = \overline{b} - b$ so that the new constraint has right-hand side $b + \Delta b$. If the basic variables are kept the same, then the final tableau changes to

$$\begin{array}{cc|c} B^{-1}A & B^{-1} & B^{-1}(b+\Delta b) \\ \hline c^T - c_B^T B^{-1}A & -c_B^T B^{-1} & -c_B^T B^{-1}(b+\Delta b) \end{array}.$$

The only way this tableau can fail to be a final tableau is if $B^{-1}(b+\Delta b) \geq 0$ is violated. Let e_j be the $m \times 1$ matrix with all entries equal to zero except in the $j'th$ row where the entry is a one. Write $\Delta b = \Delta b_1 e_1 + \cdots + \Delta b_m e_m$ where $\Delta b_j = \overline{b}_j - b_j$. It follows that $B^{-1}(\Delta b) = \Delta b_1 B^{-1}(e_1) + \cdots + \Delta b_m B^{-1}(e_m)$ where $B^{-1}(e_j)$ is the $j'th$ column in B^{-1}. The requirement $B^{-1}(b+\Delta b) \geq 0$ is equivalent to a system of linear inequalities in the Δb_j.

Example

Consider the problem

$$\max x_1 + 2x_2 + x_3 \text{ when } \begin{cases} x_1 - x_2 + 2x_3 \leq 10 \\ 2x_1 + x_2 + 3x_3 \leq 12 \\ x_1, x_2, x_3 \geq 0 \end{cases}.$$

The final tableau is given by

$$\begin{array}{ccccc|c} 3 & 0 & 5 & 1 & 1 & 22 \\ 2 & 1 & 3 & 0 & 1 & 12 \\ \hline -3 & 0 & -5 & 0 & -2 & -24 \end{array}.$$

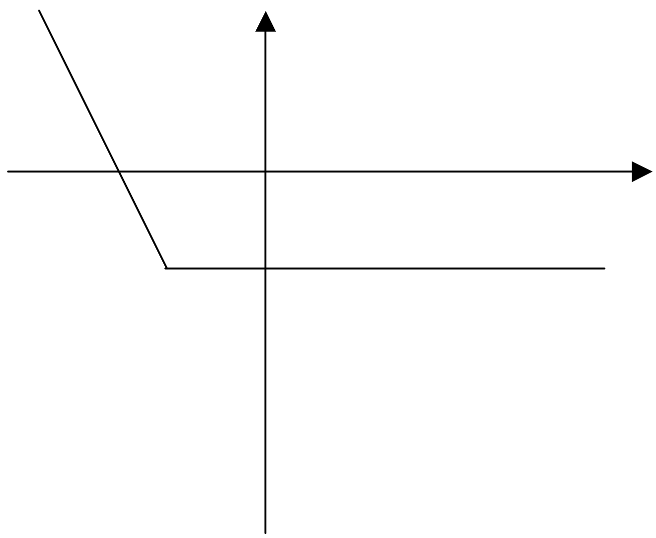
To determine how much b can vary without a change of basic variables look at the condition $B^{-1}(b+\Delta b) \geq 0$. In this example

$$B^{-1} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}.$$

It follows that

$$\begin{bmatrix} 22 \\ 12 \end{bmatrix} + \Delta b_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \Delta b_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} \geq \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

From this it follows that $\Delta b_2 \geq -12$. The admissible region looks like:



Nothing is sensitive. Suppose $\Delta b_1 = -7$ and $\Delta b_2 = -8$. It follows that

$$\overline{b} = \begin{bmatrix} 3 \\ 4 \end{bmatrix},$$

and

$$B^{-1}\overline{b} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 3 \\ 4 \end{bmatrix} = \begin{bmatrix} 7 \\ 4 \end{bmatrix}.$$

The new optimal solution is given by $x_1 = 0, x_2 = 4, x_3 = 0$. A quick check shows that this solution is in the new feasible set. The new value of the primal objective value is equal to the new value of the dual objective.

10.3 Buying resources.

Interpret b as a limit on the available resources. Suppose the cost, in dollars per unit, of resource j is given by α_j. Assume that there is an additional amount of β dollars to buy more resources. Let β_j be the amount spent on resource j. It is natural to ask how the additional money should be spent. The change in resource j is given by $\Delta b_j = \beta_j / \alpha_j$. The change in objective value is given by

$$\frac{\beta_1}{\alpha_1} c_B^T B^{-1}(e_1) + \cdots + \frac{\beta_m}{\alpha_m} c_B^T B^{-1}(e_m).$$

Maximize this function when $\beta_1 + \cdots + \beta_m \leq \beta$ and $\beta_j \geq 0$ for all j. This is an example of a linear programming problem in primal form with initial tableau

$$\begin{array}{cccc|c} 1 & \cdots & 1 & 1 & \beta \\ \hline \dfrac{c_B^T B^{-1}(e_1)}{\alpha_1} & \cdots & \dfrac{c_B^T B^{-1}(e_m)}{\alpha_m} & 0 & 0 \end{array}.$$

The next step in the simplex algorithm leads to a final tableau. The conclusion is that all money should be spent on the resource k with

$$\frac{c_B^T B^{-1}(e_k)}{\alpha_k} = \max_{j=1,\ldots,m} \left\{ \frac{c_B^T B^{-1}(e_j)}{\alpha_j} \right\},$$

provided that $B^{-1}\overline{b} \geq 0$ where

$$\overline{b} = b + \frac{\beta}{\alpha_k} e_k.$$

10.4 Variation in non-basic variable cost.

Let $\overline{c} \in \mathbb{R}^n$ be the new cost. Define $\Delta c = \overline{c} - c$ and $\Delta c_{BN} = \overline{c}_{BN} - c_{BN}$. Define $\Delta c_B \in \mathbb{R}^m$ by choosing the entries in Δc_{BN} corresponding to the basic variables in the appropriate order. The final tableau

$$\begin{array}{cc|c} B^{-1}A & B^{-1} & B^{-1}b \\ \hline c^T - c_B^T B^{-1}A & -c_B^T B^{-1} & -c_B^T B^{-1}b \end{array},$$

changes to

$$\begin{array}{cc|c} B^{-1}A & B^{-1} & B^{-1}b \\ \hline (c + \Delta c)^T - (c_B + \Delta c_B)^T B^{-1}A & -(c_B + \Delta c_B)^T B^{-1} & -(c_B + \Delta c_B)^T B^{-1}b \end{array}.$$

This time the optimality condition, as it is expressed in terms of the bottom row, leads to restrictions on Δc.

In the special case $\Delta c_B = 0$ the analysis simplifies considerably. In this case the only limiting inequality is given by $\Delta c^T + \left(c^T - c_B^T B^{-1}A\right) \leq 0$. In terms of the slack in the dual problem this inequality is equivalent to $\Delta c \leq \hat{\mu}$. There is no change in the optimal value in this case.

Example

Consider the problem

$$\max x_1 + 2x_2 + x_3 \text{ when } \begin{cases} x_1 - x_2 + 2x_3 \leq 10 \\ 2x_1 + x_2 + 3x_3 \leq 12 \\ x_1, x_2, x_3 \geq 0 \end{cases}.$$

The final tableau is given by

3	0	5	1	1	22
2	1	3	0	1	12
−3	0	−5	0	−2	−24

.

Deduce that $\hat{\mu}^T = \begin{bmatrix} 3 & 0 & 5 \end{bmatrix}$ and $\Delta c^T \le \begin{bmatrix} 3 & 0 & 5 \end{bmatrix}$, with $(\Delta c)_2 = 0$, will preserve the optimality conditions. To test the limits, change the primal objective to $4x_1 + 2x_2 + 6x_3$. The dual constraints

$$\begin{aligned} &\lambda_1 + 2\lambda_2 \ge 4 \\ &-\lambda_1 + \lambda_2 \ge 2 \ , \\ &2\lambda_1 + 3\lambda_2 \ge 6 \end{aligned}$$

are satisfied with no slack by the dual solution $\hat{\lambda}^T = \begin{bmatrix} 0 & 2 \end{bmatrix}$. Conversely, change the primal objective to $5x_1 + 2x_2 + x_3$. The optimal primal solution changes from $\hat{x}^T = \begin{bmatrix} 0 & 12 & 0 \end{bmatrix}$ to $\hat{x}^T = \begin{bmatrix} 6 & 0 & 0 \end{bmatrix}$ with dual solution $\hat{\lambda}^T = \begin{bmatrix} 0 & 5/2 \end{bmatrix}$. A quick application of the Verification Theorem proves this. It is also clear that the previous optimal solution is no longer optimal.

10.5 Variation in basic variable cost.

In this case the entire bottom row of the tableau is affected. Both

$$(c + \Delta c)^T - (c_B + \Delta c_B)^T B^{-1} A \le 0 \,,$$

and

$$-(c_B + \Delta c_B)^T B^{-1} \le 0$$

must hold.

Example

Consider the problem

$$\max 80x_1 + 70x_2 \ \text{ when } \begin{cases} 6x_1 + 3x_2 \le 96 \\ x_1 + x_2 \le 18 \\ 2x_1 + 6x_2 \le 72 \\ x_1, x_2 \ge 0 \end{cases}.$$

The final tableau is given by

1	0	1/3	−1	0	14
0	1	−1/3	2	0	4
0	0	4/3	−10	1	20
0	0	−10/3	−60	0	−1400

.

Rewrite the optimality inequalities as $-\Delta c_B^T B^{-1} \le c_B^T B^{-1}$ and $\Delta c^T - \Delta c_B^T B^{-1} A \le c_B^T B^{-1} A - c^T$. In the current example they correspond to

$$-\begin{bmatrix} \Delta c_1 & \Delta c_2 & 0 \end{bmatrix} \begin{bmatrix} 1/3 & -1 & 0 \\ -1/3 & 2 & 0 \\ 4/3 & -10 & 1 \end{bmatrix} \le \begin{bmatrix} 10/3 & 60 & 0 \end{bmatrix},$$

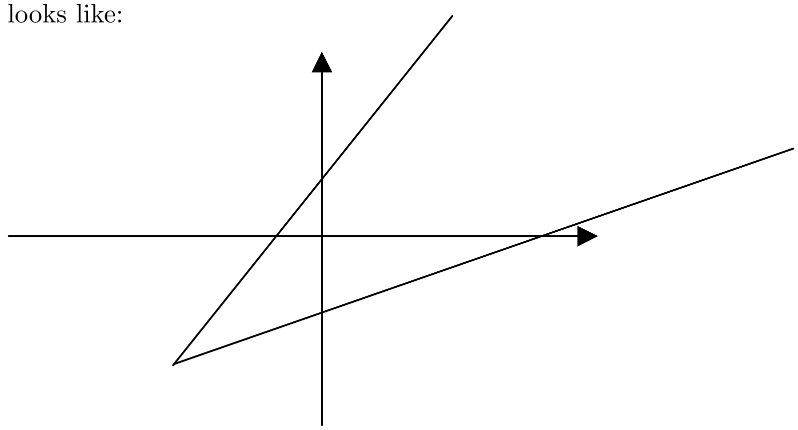
and

$$\begin{bmatrix}\Delta c_1 & \Delta c_2\end{bmatrix} - \begin{bmatrix}\Delta c_1 & \Delta c_2 & 0\end{bmatrix}\begin{bmatrix}1 & 0\\ 0 & 1\\ 0 & 0\end{bmatrix} \le \begin{bmatrix}0 & 0\end{bmatrix}.$$

The last matrix inequality is trivially satisfied. The first matrix inequality is equivalent to

$$\begin{aligned} -\Delta c_1 + \Delta c_2 &\le 10 \\ \Delta c_1 - 2\Delta c_2 &\le 60 \end{aligned}.$$

The admissible region looks like:



There is an alternative way to derive the same relationship. Since the problem only involves two variables, graph the feasible set. The solution is at the vertex $(14,4)$, which is at the intersection of the lines $6x_1 + 3x_2 = 96$ and $x_1 + x_2 = 18$. The slope is -2 and -1, respectively. To avoid changing the optimal point, the slope of the level lines corresponding to the objective $(80 + \Delta c_1)x_1 + (70 + \Delta c_2)x_2$ must satisfy

$$-2 \le -\frac{(80 + \Delta c_1)}{(70 + \Delta c_2)} \le -1.$$

This is equivalent to $70 + \Delta c_2 \le 80 + \Delta c_1 \le 140 + 2\Delta c_2$, which is equivalent to the previous pair of inequalities.

10.6 Variations in efficiency.

The tableau

$B^{-1}A$	B^{-1}	$B^{-1}b$
$c^T - c_B^T B^{-1}A$	$-c_B^T B^{-1}$	$-c_B^T B^{-1}b$

reveals that variations in the left-hand side of the constraints impact the entire tableau. The reason for this is that the matrix B depends on the matrix A. It is cumbersome to track the propagation of variations in A into the matrix B^{-1}. From a practical point of view it is comforting to know that the entries in A often are related to hardware and technology as opposed to market supply or demand. It follows that the entries in A are less susceptible to variations. In large-scale problems a different concern is the possibility of data entry errors. In a problem with 1,000 variables and 100 constraints the matrix A has 100,000 entries. Granted that many entries likely are zero, it is still possible to 'misplace' or otherwise supply wrong numbers.

10.7 Additional products.

A manufacturer is currently operating as the optimal solution dictates in the primal problem

$$\max c^T x \text{ when } \begin{cases} Ax \le b \\ x \ge 0 \end{cases}.$$

Suppose the manufacturer contemplates the introduction of a new product. Assume that no new resources are added. Let x_{n+1} represent the level of production of the new product. Write the resulting constraints as

$$\begin{bmatrix} A & \begin{matrix} \alpha_1 \\ \vdots \\ \alpha_m \end{matrix} \end{bmatrix} \begin{bmatrix} x \\ x_{n+1} \end{bmatrix} \le b, \ \begin{bmatrix} x \\ x_{n+1} \end{bmatrix} \ge \begin{bmatrix} 0 \\ 0 \end{bmatrix},$$

where α_j is the amount of resource j used when producing one unit of the new product. The new primal objective is given by

$$\begin{bmatrix} c^T & c_{n+1} \end{bmatrix} \begin{bmatrix} x \\ x_{n+1} \end{bmatrix}.$$

Suppose that $\alpha_1 \hat{\lambda}_1 + \cdots + \alpha_m \hat{\lambda}_m < c_{n+1}$ for all optimal dual solutions $\hat{\lambda}$ in the old setting. Assume the new optimal primal solution is

$$\begin{bmatrix} \hat{x} \\ 0 \end{bmatrix}.$$

Since

$$\begin{bmatrix} A & \begin{matrix} \alpha_1 \\ \vdots \\ \alpha_m \end{matrix} \end{bmatrix} \begin{bmatrix} \hat{x} \\ 0 \end{bmatrix} \le b, \ \begin{bmatrix} \hat{x} \\ 0 \end{bmatrix} \ge \begin{bmatrix} 0 \\ 0 \end{bmatrix},$$

it follows that $A\hat{x} \le b, \hat{x} \ge 0$ and hence $\hat{x}$ is feasible in the original setting. Similarly, if x is feasible in the old setting, then

$$\begin{bmatrix} x \\ 0 \end{bmatrix}$$

is feasible in the new setting. Since

$$c^T \hat{x} = \begin{bmatrix} c^T & c_{n+1} \end{bmatrix} \begin{bmatrix} \hat{x} \\ 0 \end{bmatrix} \ge \begin{bmatrix} c^T & c_{n+1} \end{bmatrix} \begin{bmatrix} x \\ x_{n+1} \end{bmatrix}$$

for all feasible points in the new setting, it must be that

$$c^T \hat{x} \ge \begin{bmatrix} c^T & c_{n+1} \end{bmatrix} \begin{bmatrix} x \\ 0 \end{bmatrix} = c^T x ,$$

and hence $\hat{x}$ is optimal in the old setting. By the Fundamental Theorem, each optimal dual solution $\hat{\lambda}$ in the old setting must satisfy $c^T \hat{x} = \hat{\lambda}^T b$. Since $\alpha_1 \hat{\lambda}_1 + \cdots + \alpha_m \hat{\lambda}_m < c_{n+1}$ holds, no $\hat{\lambda}$ is feasible in the new setting. Let $\overline{\lambda}$ be optimal in the dual problem of the new setting. The constraints

$$\begin{bmatrix} A^T \\ \alpha_1 \cdots \alpha_m \end{bmatrix} \bar{\lambda} \geq \begin{bmatrix} c \\ c_{n+1} \end{bmatrix}, \bar{\lambda} \geq 0$$

are satisfied. It follows that $A^T \bar{\lambda} \geq c, \bar{\lambda} \geq 0$ so $\bar{\lambda}$ is feasible in the dual of the old setting. By the Fundamental Theorem the objectives are equal in the new setting and hence

$$c^T \hat{x} = \begin{bmatrix} c^T & c_{n+1} \end{bmatrix} \begin{bmatrix} \hat{x} \\ 0 \end{bmatrix} = \bar{\lambda}^T b\,.$$

By the Verification Theorem, the solution $\bar{\lambda}$ is optimal in the dual of the old setting. This is a contradiction since none of the optimal solutions in the dual of the old setting are feasible in the new setting. Hence, the introduction of a new product should only be considered if

$$c_{n+1} > \alpha_1 \hat{\lambda}_1 + \cdots + \alpha_m \hat{\lambda}_m$$

for all optimal solutions in the dual of the old setting.

Example

A manufacturer produces a standard cage in a two-step process. In the first step the cage is machine-assembled. It takes four hours to assemble one cage. In the second step the cage is hand-painted. It takes two hours to paint one cage. The manufacturer has rented the assembly machine for 120 hours each week. A paint crew is hired and works 60 hours each week. The manufacturer considers introducing a collapsible cage that takes ten hours to assemble and nine hours to paint. Each standard cage built is sold for \$200 and each collapsible cage is sold for \$800. The initial tableau in the original setting is

$$\begin{array}{ccc|c} 4 & 1 & 0 & 120 \\ 2 & 0 & 1 & 60 \\ \hline 200 & 0 & 0 & 0 \end{array}\,.$$

The final tableau in the original setting is

$$\begin{array}{ccc|c} 1 & 1/4 & 0 & 30 \\ 0 & -1/2 & 1 & 0 \\ \hline 0 & -50 & 0 & -6000 \end{array}\,.$$

An optimal solution to the dual in the original setting is given by $\hat{\lambda}^T = \begin{bmatrix} 50 & 0 \end{bmatrix}$. This solution is not feasible in the new setting since $10\hat{\lambda}_1 + 9\hat{\lambda}_2 = 500 < 800$. To conclude that the new product should be introduced is a mistake. Since the ratios in the initial tableau are equal, it is possible to pivot on the second row and get

$$\begin{array}{ccc|c} 0 & 1 & -2 & 0 \\ 1 & 0 & 1/2 & 30 \\ \hline 0 & 0 & -100 & -6000 \end{array}\,.$$

This time $\hat{\lambda}^T = \begin{bmatrix} 0 & 100 \end{bmatrix}$. This solution is feasible in the new setting since

$10\hat{\lambda}_1 + 9\hat{\lambda}_2 = 900 \geq 800$. The Verification Theorem guarantees that $\hat{x}_1 = 30, \hat{x}_2 = 0$ is optimal since both objectives have value 6000. The new setting has initial tableau

$$\begin{array}{cccc|c} 4 & 10 & 1 & 0 & 120 \\ 2 & 9 & 0 & 1 & 60 \\ \hline 200 & 800 & 0 & 0 & 0 \end{array} .$$

The next tableau is given by

$$\begin{array}{cccc|c} 16/9 & 0 & 1 & -10/9 & 160/3 \\ 2/9 & 1 & 0 & 1/9 & 20/3 \\ \hline 200/9 & 0 & 0 & -800/9 & -16000/3 \end{array} .$$

The final tableau is given by

$$\begin{array}{cccc|c} 0 & -8 & 1 & -2 & 0 \\ 1 & 9/2 & 0 & 1/2 & 30 \\ \hline 0 & -100 & 0 & -100 & -6000 \end{array} ,$$

or

$$\begin{array}{cccc|c} 1 & 0 & 9/16 & -5/8 & 30 \\ 0 & 1 & -1/8 & 1/4 & 0 \\ \hline 0 & 0 & -25/2 & -225/3 & -6000 \end{array} .$$

Observe how the same optimal primal solution corresponds to two different dual solutions. This example illustrates some of the difficulties encountered in the presence of *degeneracy*, i.e., when one of more of the *basic* variables have value zero.