12 Game theory and linear programming.

12.1 Positive payoff matrix.

A general payoff matrix may have entries that are zero, positive or negative. Assume that all entries are positive. In order to conform to the standard setup of the primal problem, arrange so that the columns of the payoff matrix represent the plans of the player that prefers low values. Suppose there are n plans. Each time the game is played the strategy is to choose a plan according to a fixed probability distribution $\{p_1,\ldots,p_n\}$. Assume the opponent chooses from m plans. Let the payoff matrix be given by the $m\times n$ matrix A with entries a_j^i such that $i\in\{1,\ldots,n\}$ and $j\in\{1,\ldots,m\}$. The player seeks a probability distribution, which *guarantees* an expected return of v with v as low as possible. Since the opponent may select a pure strategy, there are m inequalities that must be satisfied. For each $j\in\{1,\ldots,m\}$ the following must hold $a_j^1 p_1+\cdots+a_j^n p_n\le v$. It is also true that $p_i\ge 0$ for all $i\in\{1,\ldots,n\}$, and $p_1+\cdots+p_n=1$. Suppose there is a $\hat{v}>0$, such that $\hat{v}\le v$ for all $v>0$, which satisfy the m inequalities, then $\frac{1}{\hat{v}}\ge\frac{1}{v}$. Note that $\frac{1}{v}=\frac{p_1+\cdots p_n}{v}=\frac{p_1}{v}+\cdots+\frac{p_n}{v}$, and $a_j^1 p_1+\cdots+a_j^n p_n\le v$ is equivalent to

$$a_j^1\frac{p_1}{v}+\cdots+a_j^n\frac{p_n}{v}\le 1 .$$

Let $c\in R^n$ be given by

$$c=\begin{bmatrix}1\\ \vdots\\ 1\end{bmatrix}_{n\times 1} .$$

Let $b\in\mathbb{R}^m$ be given by

$$b=\begin{bmatrix}1\\ \vdots\\ 1\end{bmatrix}_{m\times 1} .$$

Let $x\in\mathbb{R}^n$ be given by

$$x=\begin{bmatrix}\frac{p_1}{v}\\ \vdots\\ \frac{p_n}{v}\end{bmatrix}_{n\times 1}\ge\begin{bmatrix}0\\ \vdots\\ 0\end{bmatrix}_{n\times 1} .$$

This converts the problem to a standard primal problem

$$\max c^T x \text{ when } \begin{cases}Ax\le b\\ x\ge 0\end{cases} .$$

12.2 Arbitrary payoff matrix.

Suppose the payoff matrix is $A = \left[a_j^i\right]$. Given $\alpha \in R$ let $A_\alpha = \left[a_j^i + \alpha\right]$. If α is sufficiently large, then A_α has only positive entries. In such a case there is a probability distribution $p \in \mathbb{R}^n$ and an expected value v_α such that

$$A_\alpha p \leq \begin{bmatrix} v_\alpha \\ \vdots \\ v_\alpha \end{bmatrix}_{m \times 1},$$

and $v_\alpha \leq v$ for all $v > 0$. From this it follows that

$$Ap = A_\alpha p - \begin{bmatrix} \alpha \\ \vdots \\ \alpha \end{bmatrix}_{m \times 1} \leq \begin{bmatrix} v_\alpha - \alpha \\ \vdots \\ v_\alpha - \alpha \end{bmatrix}_{m \times 1},$$

and $v_\alpha - \alpha \leq v - \alpha$ for all $v > 0$. Let $\hat{v} = v_\alpha - \alpha$ and observe that

$$Ap \leq \begin{bmatrix} \hat{v} \\ \vdots \\ \hat{v} \end{bmatrix}_{m \times 1},$$

and $\hat{v} \leq v$ for all $v > -\alpha$. Since each strategy associated with A must yield an expected value bigger than $-\alpha$, all strategies with a fixed probability distribution do worse than p.

12.3 The dual problem.

It is possible to mimic the previous analysis in the case of a player that prefers a large expected value. If the same payoff matrix is used, then the first step is to transpose the matrix. The problem is to find a probability distribution $q \in \mathbb{R}^m$, which maximizes v subject to $A^T q \geq v$. This time, minimize $1/v$, and everything else is done analogously.

A linear programming problems, which have the special form

$$c = \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix}_{n \times 1}, b = \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix}_{m \times 1}$$

and $A > 0$, may be interpreted as a game. The solution of the primal problem is used to determine the probability distribution for the player that prefers lower expected values. The dual solution is used to find the probability distribution for the opponent.

12.4 Example.

Suppose the payoff matrix is given by

$$\begin{bmatrix} 2 & 3 & 4 & 2 & 3 & 3 \\ 2 & 1 & 1 & 1 & 4 & 3 \end{bmatrix}.$$

Eliminate the dominated strategies first. Take the transpose so that the columns correspond to the player that prefers lower expected values. The primal problem is given by

$$\max x_1 + x_2 \text{ when } \begin{cases} 4x_1 + x_2 \leq 1 \\ 3x_1 + 4x_2 \leq 1 \\ x_1 \geq 0, x_2 \geq 0 \end{cases}.$$

The simplex method yields the following tableaux

$$(1) \begin{array}{cccc|c} 4 & 1 & 1 & 0 & 1 \\ 3 & 4 & 0 & 1 & 1 \\ \hline 1 & 1 & 0 & 0 & 0 \end{array} \quad (2) \begin{array}{cccc|c} 1 & \frac{1}{4} & \frac{1}{4} & 0 & \frac{1}{4} \\ 0 & \frac{13}{4} & -\frac{3}{4} & 1 & \frac{1}{4} \\ \hline 0 & \frac{3}{4} & -\frac{1}{4} & 0 & -\frac{1}{4} \end{array} \quad (3) \begin{array}{cccc|c} 1 & 0 & \frac{4}{13} & -\frac{1}{13} & \frac{3}{13} \\ 0 & 1 & -\frac{3}{13} & \frac{4}{13} & \frac{1}{13} \\ \hline 0 & 0 & -\frac{1}{13} & -\frac{3}{13} & -\frac{4}{13} \end{array}$$

It follows that $\frac{1}{\hat{v}} = \frac{4}{13}$, and hence $\hat{v} = \frac{13}{4}$. Since $\hat{x}_1 = \frac{3}{13}$ and $\hat{x}_2 = \frac{1}{13}$, the probability distribution is given by $\hat{p}_1 = \hat{v}\hat{x}_1 = \frac{3}{4}$ and $\hat{p}_2 = \hat{v}\hat{x}_2 = \frac{1}{4}$. Similarly, the dual solution is given by $\hat{\lambda}_1 = \frac{1}{13}$ and $\hat{\lambda}_2 = \frac{3}{13}$. The probability distribution is given by $\hat{q}_1 = \hat{v}\hat{\lambda}_1 = \frac{1}{4}$ and $\hat{q}_2 = \hat{v}\hat{\lambda}_2 = \frac{3}{4}$.