1. Use the simplex algorithm to solve the problem:

$$\begin{align*}
\text{max } & 2x_1 + 3x_2 + 6x_3 \\
\text{subject to } & \begin{cases}
2x_1 + 3x_2 + x_3 \leq 10 \\
x_1 + x_2 + 2x_3 \leq 8 \\
2x_2 + 3x_3 \leq 6 \\
x_1, x_2, x_3 \geq 0
\end{cases}
\end{align*}$$

Make sure that each step employed is the step suggested by the simplex algorithm.

The initial tableau is given by:

$$\begin{array}{cccccc|c}
2 & 3 & 1 & 1 & 0 & 0 & 10 \\
1 & 1 & 2 & 0 & 1 & 0 & 8 \\
0 & 2 & 3 & 0 & 0 & 1 & 6 \\
\hline
2 & 3 & 6 & 0 & 0 & 0 & 0
\end{array}$$

The next tableau is given by:

$$\begin{array}{cccccc|c}
2 & 4/3 & 0 & 1 & 0 & -1/3 & 8 \\
1 & -1/3 & 0 & 0 & 1 & -2/3 & 4 \\
0 & 2/3 & 1 & 0 & 0 & 1/3 & 2 \\
\hline
2 & -1 & 0 & 0 & 0 & -2 & -12
\end{array}$$

The final tableau is given by:

$$\begin{array}{cccccc|c}
0 & 2 & 0 & 1 & -2 & 1 & 0 \\
1 & -1/3 & 0 & 0 & 1 & -2/3 & 4 \\
0 & 2/3 & 1 & 0 & 0 & 1/3 & 2 \\
\hline
0 & -1/3 & 0 & 0 & -2 & -2/3 & -20
\end{array}$$

The solution is given by $\hat{x}_1 = 4, \hat{x}_2 = 0, \hat{x}_3 = 2$.

2. Convert the following problem to the form of a standard dual problem. Apply the simplex method to the corresponding primal problem. Get the minimizing $\lambda^T = [\lambda_1 \lambda_2 \lambda_3]$ from the final tableau.

The converted problem is given by:

$$\begin{align*}
\text{min } & 18\lambda_1 + 20\lambda_2 + 2\lambda_3 \\
\text{subject to } & \begin{cases}
-3\lambda_1 + 5\lambda_2 + 2\lambda_3 \geq -4 \\
6\lambda_1 - 8\lambda_3 \geq 9 \\
\lambda_1, \lambda_2, \lambda_3 \geq 0
\end{cases}
\end{align*}$$

The corresponding primal problem is given by
The initial tableau is given by
\[
\begin{array}{ccccc|c}
3 & 6 & 1 & 0 & 0 & 18 \\
5 & 0 & 0 & 1 & 0 & 20 \\
2 & -8 & 0 & 0 & 1 & 2 \\
4 & 9 & 0 & 0 & 0 & 0 \\
\end{array}
\]

The next tableau is given by
\[
\begin{array}{ccccc|c}
-1/2 & 1 & 1/6 & 0 & 0 & 3 \\
5 & 0 & 0 & 1 & 0 & 20 \\
-2 & 0 & 4/3 & 0 & 1 & 26 \\
1/2 & 0 & -3/2 & 0 & 0 & -27 \\
\end{array}
\]

The final tableau is given by
\[
\begin{array}{ccccc|c}
0 & 1 & 1/6 & 1/10 & 0 & 5 \\
1 & 0 & 0 & 1/5 & 0 & 4 \\
0 & 0 & 4/3 & 2/5 & 1 & 34 \\
0 & 0 & -3/2 & -1/10 & 0 & -29 \\
\end{array}
\]

The solution is given by $\hat{x}_1 = 3/2, \hat{x}_2 = 1/10, \hat{x}_3 = 0$.

3. **Complete** the first phase of the two-phase method to determine a feasible point of a problem with constraints:
\[
\begin{align*}
x_1 - x_2 + x_3 & \leq -1 \\
-2x_1 + x_2 + x_3 & \leq -2 \\
x_1, x_2, x_3 & \geq 0
\end{align*}
\]

Make sure that each step employed is the step suggested by the first phase of the two-phase method.

The augmented system is given by
\[
\begin{align*}
x_1 - x_2 + x_3 + y_1 &= -1 \\
-2x_1 + x_2 + x_3 + y_2 &= -2
\end{align*}
\]

The auxiliary system is given by
\[
\begin{align*}
-x_1 + x_2 - x_3 - y_1 + z_1 &= 1 \\
2x_1 - x_2 - x_3 - y_2 + z_2 &= 2
\end{align*}
\]
The initial tableau is given by
\[
\begin{array}{cccccc|c}
1 & 1 & 1 & 0 & 1 & 0 & 1 \\
2 & -1 & -1 & 0 & -1 & 0 & 1 \\
0 & 0 & 0 & 0 & -1 & -1 & 0 \\
1 & 0 & -2 & -1 & -1 & 0 & 0 \\
\end{array}
\]

The next tableau is given by
\[
\begin{array}{cccccc|c}
0 & 1/2 & -3/2 & -1 & -1/2 & 1 & 1/2 & 2 \\
1 & -1/2 & -1/2 & 0 & -1/2 & 0 & 1/2 & 1 \\
0 & 1/2 & -3/2 & -1 & -1/2 & 0 & -1/2 & 2 \\
\end{array}
\]

The final tableau is given by
\[
\begin{array}{ccccccc|c}
0 & 1 & -3 & -2 & -1 & 2 & 1 & 4 \\
1 & 0 & -2 & -1 & -1 & 1 & 1 & 3 \\
0 & 0 & 0 & 0 & 0 & -1 & -1 & 0 \\
\end{array}
\]

The feasible point is given by \( x_1 = 3, x_2 = 4, x_3 = 0 \).

4. The problem:
\[
\max -700x_1 - 900x_2 + 500x_3 + 1000x_4 \quad \text{when} \quad x_1, x_2, x_3, x_4 \geq 0
\]

has a maximum when \( \hat{x}_1 = 3, \hat{x}_2 = 0, \hat{x}_3 = 4, \hat{x}_4 = 8 \). Use nothing but complementary slackness to find the optimal solution \( \hat{\lambda} \) of the dual problem.

The dual problem is given by
\[
\min 6\lambda_1 + 5\lambda_2 + 4\lambda_3 + 8\lambda_4 \quad \text{when} \quad \begin{align*}
-\lambda_1 - \lambda_2 & \geq -700 \\
-\lambda_3 - \lambda_4 & \geq -900 \\
\lambda_1 + \lambda_3 & \geq 500 \\
\lambda_2 + \lambda_4 & \geq 1000 \\
\lambda_1, \lambda_2, \lambda_3, \lambda_4 & \geq 0
\end{align*}
\]

Complementary slackness: \( \hat{x}_1 > 0 \Rightarrow \hat{\lambda}_1 + \hat{\lambda}_2 = 700, \hat{x}_3 > 0 \Rightarrow \hat{\lambda}_1 + \hat{\lambda}_3 = 500, \hat{x}_4 > 0 \Rightarrow \hat{\lambda}_2 + \hat{\lambda}_4 = 1000, \hat{y}_1 > 0 \Rightarrow \hat{\lambda}_1 = 0 \).

It follows that the dual solution is given by \( \hat{\lambda}_1 = 0, \hat{\lambda}_2 = 700, \hat{\lambda}_3 = 500, \hat{\lambda}_4 = 300 \).
5. Consider the problem:

$$\begin{align*}
\text{max } x_1 + 2x_2 & \quad \text{when} \\
& \quad \begin{cases}
    x_1 \leq 10 \\
    x_2 \leq x_1 \\
    2x_1 \geq x_2 \\
    x_1 \geq 27 - x_2 \\
    x_1, x_2 \geq 0
\end{cases}
\end{align*}$$

(a) Use the **graphical method** to determine the solution.

(b) Convert the problem to standard form and use complementary slackness to get the solution to the dual problem.

The converted problem is given by

$$\begin{align*}
\text{max } x_1 + 2x_2 & \quad \text{when} \\
& \quad \begin{cases}
    x_1 \leq 10 \\
    x_1 - x_2 \leq 0 \\
    -2x_1 + x_2 \leq 0 \\
    -x_1 - x_2 \leq -27 \\
    x_1, x_2 \geq 0
\end{cases}
\end{align*}$$

The corresponding dual problem is given by

$$\begin{align*}
\text{min } 10\lambda_1 - 27\lambda_4 & \quad \text{when} \\
& \quad \begin{cases}
    \lambda_1 + \lambda_2 - 2\lambda_3 - \lambda_4 \geq 1 \\
    -\lambda_2 + \lambda_3 - \lambda_4 \geq 2 \\
    \lambda_1, \lambda_2, \lambda_3, \lambda_4 \geq 0
\end{cases}
\end{align*}$$

With $\hat{x}_1 = 10, \hat{x}_2 = 20$ there is slack in the second and fourth inequality. It follows that $\hat{\lambda}_2 = \hat{\lambda}_4 = 0$. Both inequalities in the dual problem must in fact be satisfied with equality, and it follows that $\hat{\lambda}_3 = 2$ and also $\hat{\lambda}_1 = 5$.

(c) Use the Verification Theorem to prove that your solution is indeed optimal.

Both objectives evaluate to 50 so both solutions are optimal.