

## Exam 2

1. Consider a primal problem with

$$A = \begin{bmatrix} 2 & 3 & 1 \\ 1 & 1 & 2 \\ 0 & 2 & 3 \end{bmatrix},$$

and a final tableau given by

$$\begin{array}{cccccc|c} 0 & 3 & 0 & 1 & -2 & 1 & 0 \\ 1 & -1/3 & 0 & 0 & 1 & -2/3 & 4 \\ 0 & 2/3 & 1 & 0 & 0 & 1/3 & 2 \\ \hline 0 & -1/3 & 0 & 0 & -2 & -2/3 & -20 \end{array}.$$

(a) Determine  $b$ .

The basic variables are  $y_1, x_1, x_3$  and it follows that

$$B = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 3 \end{bmatrix}.$$

Since

$$B^{-1}b = \begin{bmatrix} 0 \\ 4 \\ 2 \end{bmatrix},$$

it must be that

$$b = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} 0 \\ 4 \\ 2 \end{bmatrix} = \begin{bmatrix} 10 \\ 8 \\ 6 \end{bmatrix}.$$

(b) Determine  $c$ .

Since  $c^T - c_B^T B^{-1}A = [0 \ -1/3 \ 0]$  and  $c_B^T B^{-1} = [0 \ 2 \ 2/3]$ , it follows that

$$c^T = [0 \ 2 \ 2/3] \begin{bmatrix} 2 & 3 & 1 \\ 1 & 1 & 2 \\ 0 & 2 & 3 \end{bmatrix} + [0 \ -1/3 \ 0] = [2 \ 10/3 \ 6] + [0 \ -1/3 \ 0] = [2 \ 3 \ 6]$$

2. Consider the problem

$$\max 3x_1 + x_2 + 3x_3 \text{ when } \begin{cases} 2x_1 + x_2 + x_3 \leq 2 \\ x_1 + 2x_2 + 3x_3 \leq 5 \\ 2x_1 + 2x_2 + x_3 \leq 6 \\ x_1, x_2, x_3 \geq 0 \end{cases}$$

with a final tableau given by

$$\begin{array}{cccccc|c} 1 & 1/5 & 0 & 3/5 & -1/5 & 0 & 1/5 \\ 0 & 3/5 & 1 & -1/5 & 2/5 & 0 & 8/5 \\ 0 & 1 & 0 & -1 & 0 & 1 & 4 \\ \hline 0 & -7/5 & 0 & -6/5 & -3/5 & 0 & -27/5 \end{array}$$

(a) The third slack variable has value 4. It follows that  $b_3$  can be reduced from 6 to 2 with the same optimal solution. Is it possible to reduce  $b_3$  by more than 4 units and still have the same basic variables in the optimal solution? If so, by how much?

Since

$$B^{-1}(b + \Delta b) = B^{-1}b + B^{-1} \begin{bmatrix} 0 \\ 0 \\ \Delta b_3 \end{bmatrix} = B^{-1}b + \begin{bmatrix} 0 \\ 0 \\ \Delta b_3 \end{bmatrix},$$

so only the third entry changes, it follows that  $4 + \Delta b_3 \geq 0$  and hence  $\Delta b_3 \geq -4$ . The answer is no.

(b) The first slack variable has value 0. Is it possible to reduce  $b_1$  and still have the same basic variables in the optimal solution? If so, by how much?

This time

$$B^{-1}(b + \Delta b) = B^{-1}b + B^{-1} \begin{bmatrix} \Delta b_1 \\ 0 \\ 0 \end{bmatrix} = B^{-1}b + \begin{bmatrix} \frac{3}{5}\Delta b_1 \\ -\frac{1}{5}\Delta b_1 \\ -\Delta b_1 \end{bmatrix},$$

and

$$\begin{aligned} \frac{1}{5} + \frac{3}{5}\Delta b_1 &\geq 0 \\ \frac{8}{5} - \frac{1}{5}\Delta b_1 &\geq 0 \\ 4 - \Delta b_1 &\geq 0 \end{aligned}$$

It follows that  $-\frac{1}{3} \leq \Delta b_1 \leq 4$ , and hence the answer is yes, by a third of a unit.

(c) The dual solution is unique. By how much should  $c_2$  increase to make it worthwhile to start producing the product represented by  $x_2$ .

With the objective  $3x_1 + c_2x_2 + 3x_3$  the second constraint in the dual changes to

$\lambda_1 + 2\lambda_2 + 2\lambda_3 \geq c_2$ . With the current dual solution this gives  $\frac{6}{5} + 2\frac{3}{5} + 0 = \frac{12}{5} \geq c_2$ . The increase must be bigger than  $\frac{12}{5} - 1 = \frac{7}{5}$ .

3. Consider the problem

$$\max 3x_1 + x_2 + 3x_3 \text{ when } \begin{cases} 2x_1 + x_2 + x_3 \leq 2 \\ x_1 + 2x_2 + 3x_3 \leq 5 \\ 2x_1 + 2x_2 + x_3 \leq 6 \\ x_1, x_2, x_3 \geq 0 \end{cases}$$

with a final tableau given by

$$\begin{array}{cccccc|c} 1 & 1/5 & 0 & 3/5 & -1/5 & 0 & 1/5 \\ 0 & 3/5 & 1 & -1/5 & 2/5 & 0 & 8/5 \\ 0 & 1 & 0 & -1 & 0 & 1 & 4 \\ \hline 0 & -7/5 & 0 & -6/5 & -3/5 & 0 & -27/5 \end{array}$$

(a) What is the admissible range of variation in  $c_2$  that retains the same optimal solution?

Since  $x_2$  is a non-basic variable the only concern is the inequality

$$(c^T + [0 \ \Delta c_2 \ 0]) - c_B^T B^{-1} A \leq [0 \ 0 \ 0].$$

This simplifies to  $[0 \ \Delta c_2 \ 0] \leq [0 \ 7/5 \ 0]$ . In other words,  $c_2$  can be increased by no more than  $7/5$ .

(b) Determine the admissible region for  $\Delta c_1, \Delta c_3$  that retains the same optimal solution.

Both variables are basic and the system of inequalities

$$\begin{aligned} (c^T + [\Delta c_1 \ 0 \ \Delta c_3]) - (c_B^T + [\Delta c_1 \ \Delta c_3 \ 0]) B^{-1} A &\leq [0 \ 0 \ 0] \\ -(c_B^T + [\Delta c_1 \ \Delta c_3 \ 0]) B^{-1} &\leq [0 \ 0 \ 0] \end{aligned}$$

must be satisfied. This simplifies as

$$\begin{aligned} [\Delta c_1 \ 0 \ \Delta c_3] - [\Delta c_1 \ \Delta c_3 \ 0] \begin{bmatrix} 1 & 1/5 & 0 \\ 0 & 3/5 & 1 \\ 0 & 1 & 0 \end{bmatrix} &\leq [0 \ 7/5 \ 0] \\ -[\Delta c_1 \ \Delta c_3 \ 0] \begin{bmatrix} 3/5 & -1/5 & 0 \\ -1/5 & 2/5 & 0 \\ -1 & 0 & 1 \end{bmatrix} &\leq [6/5 \ 3/5 \ 0] \end{aligned}$$

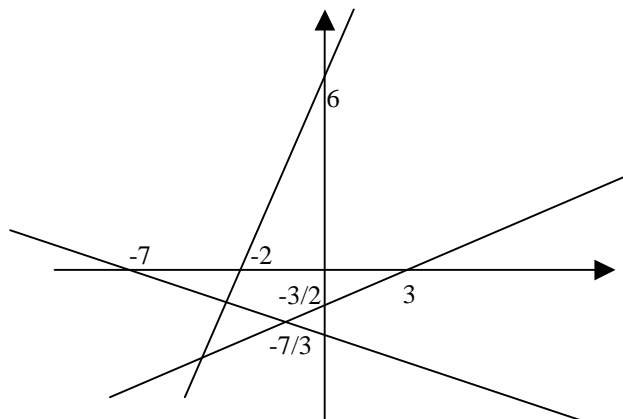
or

$$\begin{aligned} [\Delta c_1 \ 0 \ \Delta c_3] - [\Delta c_1 \ \frac{1}{5}\Delta c_1 + \frac{3}{5}\Delta c_3 \ \Delta c_3] &\leq [0 \ 7/5 \ 0] \\ [-\frac{3}{5}\Delta c_1 + \frac{1}{5}\Delta c_3 \ \frac{1}{5}\Delta c_1 - \frac{2}{5}\Delta c_3 \ 0] &\leq [6/5 \ 3/5 \ 0] \end{aligned}$$

The admissible region is given by

$$\begin{aligned} -\Delta c_1 - 3\Delta c_3 &\leq 7 \\ -3\Delta c_1 + \Delta c_3 &\leq 6 \\ \Delta c_1 - 2\Delta c_3 &\leq 3 \end{aligned}$$

(c) Neither  $c_1$  nor  $c_3$  is sensitive.

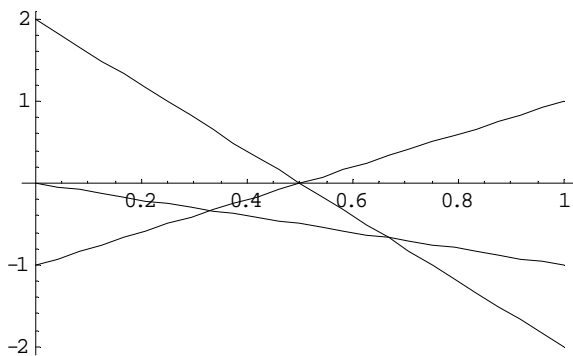


Let the payoff matrix of a game be given by

$$\begin{bmatrix} 1 & 2 & -2 & -1 & 2 & 1 \\ -1 & 0 & -1 & 0 & 1 & -1 \\ 1 & 0 & 2 & 0 & 1 & -1 \end{bmatrix},$$

(a) Determine both players' optimal strategy.

The row-player's third plan dominates the second plan. The column player eliminates options 1,2 and 5 since they are dominated by option 6. Assume the row-player chooses option 1 with probability  $x$ . The expected payoffs for the column-player are  $-2x + 2(1 - x) = 2 - 4x$ ,  $-x$ ,  $2x - 1$  for plan 3,4,6 respectively.



The best worst-case scenario is if the row-player chooses  $x$  such that  $-x = 2x - 1$ , i.e.,  $x = 1/3$ . The row-player should play according to  $[1/3 \ 0 \ 2/3]$  and the value of the game is  $-1/3$ .

(b) Determine the column-player's optimal strategy.

The column-player only considers plan 4 and 6. Let  $y$  be the probability to use option 4. It must be that  $0 \cdot y - 1 \cdot (1 - y) = -1/3$  and hence  $y = 2/3$ . The column-player should play according to  $[0 \ 0 \ 0 \ 2/3 \ 0 \ 1/3]$ .

(c) Who does the game favor? The column-player.