Exam 2

1. Consider a primal problem with

\[
A = \begin{bmatrix} 2 & 3 & 1 \\ 1 & 1 & 2 \\ 0 & 2 & 3 \end{bmatrix},
\]

and a final tableau given by

\[
\begin{array}{cccccc|c}
0 & 3 & 0 & 1 & -2 & 1 & 0 \\
1 & -1/3 & 0 & 0 & 1 & -2/3 & 4 \\
0 & 2/3 & 1 & 0 & 0 & 1/3 & 2 \\
\hline
0 & -1/3 & 0 & 0 & -2 & -2/3 & -20
\end{array}
\]

(a) Determine \( b \).

The basic variables are \( y_1, x_1, x_3 \) and it follows that

\[
B = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 3 \end{bmatrix}.
\]

Since

\[
B^{-1}b = \begin{bmatrix} 0 \\ 4 \\ 2 \end{bmatrix},
\]

it must be that

\[
b = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} 0 \\ 4 \\ 2 \end{bmatrix} = \begin{bmatrix} 10 \\ 8 \\ 6 \end{bmatrix}.
\]

(b) Determine \( c \).

Since \( c^T - c_B^T B^{-1} A = \begin{bmatrix} 0 & -1/3 & 0 \end{bmatrix} \) and \( c_B^T B^{-1} = \begin{bmatrix} 0 & 2 & 2/3 \end{bmatrix} \), it follows that

\[
c^T = \begin{bmatrix} 2 & 3 & 1 \\ 0 & 2 & 2/3 \\ 1 & 1 & 2 \end{bmatrix} \begin{bmatrix} 0 & -1/3 & 0 \end{bmatrix} = \begin{bmatrix} 10/3 & 6 \\ 1 & 1/3 \end{bmatrix} = \begin{bmatrix} 2 & 3 & 6 \end{bmatrix}
\]
Consider the problem

\[
\begin{align*}
\text{max } 3x_1 + x_2 + 3x_3 \quad \text{when} \\
& \quad \begin{cases}
2x_1 + x_2 + x_3 \leq 2 \\
\frac{x_1}{2} + 2x_2 + 3x_3 \leq 5 \\
2x_1 \geq 2x_2 + x_3 \leq 6' \\
x_1, x_2, x_3 \geq 0 
\end{cases}
\end{align*}
\]

with a final tableau given by

\[
\begin{array}{cccccccc|c}
1 & 1/5 & 0 & 3/5 & -1/5 & 0 & 1/5 \\
0 & 3/5 & 1 & -1/5 & 2/5 & 0 & 8/5 \\
0 & 1 & 0 & -1 & 0 & 1 & 4 \\
\hline
0 & -7/5 & 0 & -6/5 & -3/5 & 0 & -27/5 \\
\end{array}
\]

(a) The third slack variable has value 4. It follows that \( b_3 \) can be reduced from 6 to 2 with the same optimal solution. Is it possible to reduce \( b_3 \) by more than 4 units and still have the same basic variables in the optimal solution? If so, by how much?

Since

\[
B^{-1}(b + \Delta b) = B^{-1}b + B^{-1}\Delta b = 0
\]

so only the third entry changes, it follows that \( 4 + \Delta b_3 \geq 0 \) and hence \( \Delta b_3 \geq -4 \). The answer is no.

(b) The first slack variable has value 0. Is it possible to reduce \( b_1 \) and still have the same basic variables in the optimal solution? If so, by how much?

This time

\[
B^{-1}(b + \Delta b) = B^{-1}b + B^{-1}\Delta b = \begin{bmatrix}
\Delta b_1 \\
0
\end{bmatrix}
= B^{-1}b + \begin{bmatrix}
\frac{3}{5} \Delta b_1 \\
-\frac{1}{5} \Delta b_1
\end{bmatrix}
\]

and

\[
\frac{1}{5} + \frac{3}{5} \Delta b_1 \geq 0 \\
\frac{2}{5} - \frac{1}{5} \Delta b_1 \geq 0 \\
4 - \Delta b_1 \geq 0
\]

It follows that \( -\frac{1}{3} \leq \Delta b_1 \leq 4 \), and hence the answer is yes, by a third of a unit.

(c) The dual solution is unique. By how much should \( c_2 \) increase to make it worthwhile to start producing the product represented by \( x_2 \).

With the objective \( 3x_1 + c_2 x_2 + 3x_3 \) the second constraint in the dual changes to

\[
\lambda_1 + 2\lambda_2 + 2\lambda_3 \geq c_2.
\]

With the current dual solution this gives \( 6/5 + 2\lambda_2 = 12/5 \). The increase must be bigger than \( 12/5 - 1 = 7/5 \).
3. Consider the problem

\[
\begin{align*}
\text{max } & \quad 3x_1 + x_2 + 3x_3 \\
\text{subject to } & \quad 2x_1 + x_2 + x_3 \leq 2 \\
& \quad x_1 + 2x_2 + 3x_3 \leq 5 \\
& \quad 2x_1 + 2x_2 + x_3 \leq 6' \\
& \quad x_1, x_2, x_3 \geq 0
\end{align*}
\]

with a final tableau given by

\[
\begin{array}{cccccccc}
1 & 1/5 & 0 & 3/5 & -1/5 & 0 & 1/5 \\
0 & 3/5 & 1 & -1/5 & 2/5 & 0 & 8/5 \\
0 & 1 & 0 & -1 & 0 & 1 & 4 \\
0 & -7/5 & 0 & -6/5 & -3/5 & 0 & -27/5
\end{array}
\]

(a) What is the admissible range of variation in \( c_2 \) that retains the same optimal solution?

Since \( x_2 \) is a non-basic variable the only concern is the inequality

\[
(c^T + [0 \ A c_2 \ 0]) - c_B^T B^{-1} A \leq [0 \ 0 \ 0].
\]

This simplifies to \([0 \ A c_2 \ 0] \leq [0 \ 7/5 \ 0]\). In other words, \( c_2 \) can be increased by no more than 7/5.

(b) Determine the admissible region for \( \Delta c_2, \Delta c_3 \) that retains the same optimal solution.

Both variables are basic and the system of inequalities

\[
(c^T + [\Delta c_1 \ \Delta c_2 \ \Delta c_3]) - (c_B^T + [\Delta c_1 \ \Delta c_2 \ \Delta c_3])B^{-1} A \leq [0 \ 0 \ 0],
\]

must be satisfied. This simplifies as

\[
\begin{array}{cccccccc}
\Delta c_1 & 0 & \Delta c_3 & -[\Delta c_1 & \Delta c_3] & 0 & 1/5 & 0 \\
0 & 3/5 & 1 & 0 & 3/5 & 1 & \leq & 0 & 7/5 & 0 \\
0 & 1 & 0 & -1 & 0 & 1 & \leq & 6/5 & 3/5 & 0
\end{array}
\]

or

\[
\begin{array}{cccccccc}
\Delta c_1 & 0 & \Delta c_3 & -[\Delta c_1 \ \Delta c_2 \ \Delta c_3] & \leq & 0 & 7/5 & 0 \\
-\frac{3}{2}\Delta c_1 + \frac{1}{2}\Delta c_3 & \leq & 3/5 & 0
\end{array}
\]

The admissible region is given by

\(-\Delta c_1 - 3\Delta c_3 \leq 7
\)

\(-3\Delta c_1 + \Delta c_3 \leq 6.
\)

\(\Delta c_1 - 2\Delta c_3 \leq 3
\)

(c) Neither \( c_1 \) nor \( c_3 \) is sensitive.
Let the payoff matrix of a game be given by
\[
\begin{bmatrix}
1 & 2 & -2 & -1 & 2 & 1 \\
-1 & 0 & -1 & 0 & 1 & -1 \\
1 & 0 & 2 & 0 & 1 & -1
\end{bmatrix},
\]

(a) Determine both players' optimal strategy.
The row-player's third plan dominates the second plan. The column player eliminates options 1, 2 and 5 since they are dominated by option 6. Assume the row-player chooses option 1 with probability \( x \). The expected payoffs for the column-player are
\[
2 \cdot 2x + 2(1-x) = 2 - 4x, \quad -x, \quad 2x - 1
\]
for plan 3, 4, 6 respectively.

The best worst-case scenario is if the row-player chooses \( x \) such that
\[
-2x + 2(1-x) = -1/3
\]
which implies
\[
x = 1/3.
\]
The row-player should play according to \( \begin{bmatrix} 1/3 & 0 & 2/3 \end{bmatrix} \) and the value of the game is \(-1/3\).

(b) Determine the column-player's optimal strategy.
The column-player only considers plan 4 and 6. Let \( y \) be the probability to use option 4. It must be that
\[
0 \cdot y - 1 \cdot (1-y) = -1/3
\]
hence \( y = 2/3 \). The column-player should play according to \( \begin{bmatrix} 0 & 0 & 2/3 & 0 & 1/3 \end{bmatrix} \).

(c) Who does the game favor? The column-player.