MATH 336 **Final Exam** **Spring 2006**

Name: ______________________________

Student (Z) Number: ______________________________

There are 200 total points possible. You must show your work and justify your answers to receive credit. Clearly indicate your answers by CIRCLING them.

1. (14 pts.) Determine the solution $y(x)$ of the initial value problem $xy' = 3y + x^{-1}$, $y(1) = 2$.

2. (14 pts. each) Find the general solution $y(x)$ of the following differential equations.

   (a) $\dfrac{dy}{dx} = x + xy^2$

(b) $xy\dfrac{dy}{dx} = x^2 + xy + y^2$

(c) $(6xy - x^3 + 8y)\dfrac{dy}{dx} + (2x + 3y^2 - 3x^2y + 3\sqrt{x}) = 0$

3. (12 pts.) Use Euler's method with a step size of $h = 1/3$ to approximate $y(1)$, where $y(x)$ is the solution of the initial value problem $y'(x) = 3x^2 - y$, $y(0) = 1$.

4. (14 pts.) Use the method of undetermined coefficients to find the general solution $y(x)$ of $y'' + 4y' + 5y = e^{-3x}$

5. (14 pts.) Use the variation of parameters method to find the general solution of

$$y'' - 4y' + 4y = \sqrt{x}e^{2x}.$$

6. (20 pts.) Use Laplace transforms to solve the initial value problem

$$x'' + 2x' - 8x = e^t, \;\; x(0) = 1, \;\; x'(0) = 2.$$

7. (12 pts.) Using any valid method, determine the Laplace transform $F(s) = \mathcal{L}\{f(t)\}$, where

$$f(t) = \begin{cases} t - 1 & \text{if } 0 \leq t < 2, \\ 0 & \text{if } t \geq 2. \end{cases}$$

8. (12 pts.) Determine the inverse Laplace transform $f(t) = \mathcal{L}^{-1}\{F(s)\}$, where $F(s) = \dfrac{s-1}{s^2+6s+13}$.

9. (20 pts.) Find the general solution of the following system of differential equations. ($x = x(t)$ and $y = y(t)$ are both functions of the independent variable t.)

$$x' = x - 4y, \quad y' = -x - 2y.$$

10. (20 pts.) A mass of 3 kg is attached to a spring-dashpot system. Assume that the force of resistance (damping) is 12 newtons times the velocity in meters per second, and that the spring exerts a force of 9 newtons when the spring is stretched 1 meter. At time $t = 0$, the mass is pulled 4 meters from the equilibrium position and set in motion toward the equilibrium position with a speed of 16 meters per second.

    (a) Set up and solve an initial value problem to obtain an explicit formula for the position $x(t)$ of the mass.

    (b) Determine the time at which the mass first passes its equilibrium position.

11. (20 pts.) At time $t = 0$, a large tank contains 100 gallons of pure water. A brine solution containing 4 pounds of salt per gallon is pumped into the tank at a rate of 2 gallons per minute and the well-mixed solution is pumped out of the tank at a rate of 3 gallons per minute. Set up and solve the initial value problem to determine the amount $x(t)$ of salt in the tank at time t.

General Properties

$$\begin{aligned}
\mathcal{L}\{a\,f(t) + b\,g(t)\} &= a\,\mathcal{L}\{f(t)\} + b\,\mathcal{L}\{g(t)\}, \\
\mathcal{L}\{f'(t)\} &= s\mathcal{L}\{f(t)\} - f(0), \\
\mathcal{L}\{f''(t)\} &= s^2\mathcal{L}\{f(t)\} - sf(0) - f'(0),\ \ldots \\
\mathcal{L}\{e^{at}f(t)\} &= F(s-a) \quad \text{if } \ F(s) = \mathcal{L}\{f(t)\}, \\
\mathcal{L}\{u(t-a)f(t-a)\} &= e^{-as}F(s) \quad \text{if } \ F(s) = \mathcal{L}\{f(t)\}. \\
\mathcal{L}\{f(t) * g(t)\} &= \mathcal{L}\{f(t)\}\,\mathcal{L}\{g(t)\}.
\end{aligned}$$

Some Special Transforms

$$\begin{aligned}
\mathcal{L}\{e^{at}\} &= \frac{1}{s-a} \quad (s > a), \\
\mathcal{L}\{\cos(bt)\} &= \frac{s}{s^2+b^2} \quad (s > 0), \\
\mathcal{L}\{\sin(bt)\} &= \frac{b}{s^2+b^2} \quad (s > 0), \\
\mathcal{L}\{\cosh(bt)\} &= \frac{s}{s^2-b^2} \quad (s > |b|), \\
\mathcal{L}\{\sinh(bt)\} &= \frac{b}{s^2-b^2} \quad (s > |b|), \\
\mathcal{L}\{1\} = \frac{1}{s}, \qquad \mathcal{L}\{t\} = \frac{1}{s^2}, \qquad \mathcal{L}\{t^2\} &= \frac{2\cdot 1}{s^3}, \qquad \mathcal{L}\{t^3\} = \frac{3\cdot 2\cdot 1}{s^4} \quad (s > 0), \\
\mathcal{L}\{t^n\} &= \frac{n!}{s^{n+1}} \quad (s > 0) \text{ for any positive integer } n. \\
\mathcal{L}\{u(t-c)\} &= \frac{e^{-cs}}{s} \quad (s > 0).
\end{aligned}$$