

Why multivariate splines for PDEs ?

Refinement of tetrahedra and saddle point problems

Linear problems

Robustness of the method for singular perturbation problems

Nonlinear problems

Spline Element Method for Partial Differential Equations

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Outline

- 1 Why multivariate splines for PDEs ?
 - Motivation
 - Features of the method
 - Approximation properties
- 2 Refinement of tetrahedra and saddle point problems
 - Uniform refinement of a tetrahedron
 - Numerical methods for saddle point problems
- 3 Linear problems
- 4 Robustness of the method for singular perturbation problems
 - Fourth order singular problem
 - Darcy-Stokes equation
- 5 Nonlinear problems
 - Navier-Stokes equations

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Finite element implementation with Lagrange multipliers

The finite element method is the most widely used method for solving numerically partial differential equations

A collection of methods falls under the designation f.e.m.

Model problem

$$\begin{cases} -\Delta u = f & \text{in } \Omega \\ u = g & \text{on } \partial\Omega \end{cases}$$

where $\partial\Omega$ will denote the boundary of the bounded domain Ω and Δ denotes the Laplace operator, $\Delta = \sum_{i=1}^n \frac{\partial^2}{\partial x_i^2}$.

Green's identity

$$\int_{\partial\Omega} (-\operatorname{div} \nabla u) v \, dx = \int_{\partial\Omega} \nabla u \cdot \nabla v \, dx - \int_{\partial\Omega} \frac{\partial u}{\partial \nu} v.$$

Find u in $H_0^1(\Omega)$ such that

$$\int_{\Omega} \nabla u \cdot \nabla v = \int_{\Omega} f v, \quad \forall v \in H_0^1(\Omega),$$

Biharmonic equation

$$\begin{cases} \Delta^2 u = f & \text{in } \Omega \\ u = g & \text{on } \partial\Omega \\ \frac{\partial u}{\partial \mathbf{n}} = h & \text{in } \partial\Omega, \end{cases}$$

$$\int_{\Omega} \Delta u \Delta v \, dx = \int_{\Omega} f v, \quad \forall v \in H_0^2(\Omega), \quad (1)$$

Abstract variational problem Find $u \in V$ such that

$$a(u, v) = \langle l, v \rangle \quad \text{for all } v \in V$$

Lax Milgram lemma

Discrete approximations Find $u \in V_h$ such that

$$a(u, v) = \langle l, v \rangle \quad \text{for all } v \in V_h$$

Cea's lemma

$$\|u - u_h\|_V \leq C \min_{v \in V_h} \|u - v\|_V$$

for a constant C independent of h .

Requirements for conforming approximations

Let $k \geq 1$ and suppose Ω is bounded. Then a piecewise infinitely differentiable function $v : \bar{\Omega} \rightarrow \mathbf{R}$ belongs to $H^k(\Omega)$ if and only if $v \in C^{k-1}(\bar{\Omega})$.

Even in two dimensions, conforming finite element spaces can be **very complicated** and there are **no satisfactory answer in three dimensions**.

Nonconforming approximations are not **very popular**.

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Stokes equations Find $(\mathbf{u}, p) \in H^1(\Omega)^n \times L_0^2(\Omega)$ such that

$$\begin{cases} -\nu \Delta \mathbf{u} + \nabla p = \mathbf{f} & \text{in } \Omega \\ \operatorname{div} \mathbf{u} = 0 & \text{in } \Omega \\ \mathbf{u} = \mathbf{g} & \text{on } \partial\Omega \end{cases}$$

Linear elasticity equations Find (σ, u) in $H(\operatorname{div}, \Omega, \mathbb{S}) \times L^2(\Omega, \mathbb{R}^n)$ such that

$$\begin{cases} A\sigma = \epsilon(u) \\ \operatorname{div} \sigma = f \end{cases}$$

$u = g$ on the boundary and $A\sigma = \frac{1}{2\mu}\sigma - \frac{\lambda}{4\mu(\mu+\lambda)} \operatorname{tr} \sigma \delta$,

Equilibrium conditions are very difficult to enforce in the finite element method

G. Awanou, Spline element method for elasticity, **Coming soon**

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$$V_h = \{c \in \mathbf{R}^N, Rc = 0\},$$

$$W_h = \{c \in \mathbf{R}^N, Rc = G\}$$

The condition $a(u, v) = \langle l, v \rangle$ for all $v \in V_h$ becomes

$$K(c)d = L^T d \quad \forall d \in V_h, \text{ that is for all } d \text{ with } Rd = 0$$

$$K(c, c) + \lambda^T R = L^T.$$

$$\begin{bmatrix} K^T & R^T \\ R & 0 \end{bmatrix} \begin{bmatrix} c \\ \lambda \end{bmatrix} = \begin{bmatrix} L \\ G \end{bmatrix}$$

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Advantages of the method

- Can be applied to a wide range of PDEs in science and engineering in both two and three dimensions.
- Constraints and Smoothness are enforced exactly and there is no need to implement basis functions with the required properties. Particularly suitable for higher order PDEs.
- No inf-sup condition
- One gets in a single implementation approximations of variable order.
- The mass and stiffness matrices are assembled easily and this can be done in parallel.
- Easy implementation of p-adaptive approximation and simplicity of a posteriori error estimate

Possible Disadvantages

- Large size matrices for 3D problems and high order approximations
-

$$\begin{bmatrix} K^T & R^T \\ R & 0 \end{bmatrix} \begin{bmatrix} c \\ \lambda \end{bmatrix} = \begin{bmatrix} L \\ G \end{bmatrix} \quad \left(\begin{array}{cc} K^T & R^T \\ R & -\mu I \end{array} \right) \begin{bmatrix} c^{(l+1)} \\ \lambda^{(l+1)} \end{bmatrix} = \begin{bmatrix} L \\ G - \mu \lambda^{(l)} \end{bmatrix}$$

Computing $c^{(1)}$ from $\lambda^{(0)}$, one solves

$$\left(K^T + \frac{1}{\mu} R^T R \right) c^{(l+1)} = K^T c^{(l)} + \frac{1}{\mu} R^T G, \quad l = 1, 2, \dots$$

$$\|c - c^{(l+1)}\| \leq C\mu \|c - c^{(l)}\|$$

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Recall Cea's lemma $\|u - u_h\|_V \leq C \min_{v \in V_h} \|u - v\|_V$

Assume V_h is a spline space $S_d^r(\mathcal{T}_h)$

Bivariate splines For $d \geq 3r + 2$ and $0 \leq m \leq d$ and for $0 \leq k \leq m$.

$$|f - Qf|_{\rho, k} \leq Ch^{m+1-k} |f|_{\rho, m+1}$$

The constant C depends on the smallest angle in Δ

Trivariate splines For $0 \leq k \leq d$, $f \in W_\rho^{d+1}(\Omega)$

$$|f - Qf|_{\rho, k} \leq Ch^{d+1-k} |f|_{\rho, d+1}$$

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Bivariate splines For $d \geq 3r + 2$ and $0 \leq m \leq d$ and for $0 \leq k \leq m$.

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Non degenerate tetrahedron $\sigma_T = \frac{h_T}{\rho_T} < \infty$

Quasi-uniform tetrahedral partition $\sigma_T = \frac{h_T}{\rho_T} \leq \sigma < \infty, \forall T \in \mathcal{T}_h$

Care must be taken for uniform refinement

Tetrahedra	Sigma	Types
1	4.1815	1
8	6.8102	2
64	10.1948	4
512	16.6254	10
4096	26.8399	24

Example of uniform refinement

Tetrahedra	Sigma	Types
1	4.1815	1
8	4.1815	1
64	4.1815	1
512	4.1815	1
4096	4.1815	1

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$$\begin{pmatrix} A & L^T \\ L & 0 \end{pmatrix} \begin{pmatrix} \mathbf{c} \\ \lambda \end{pmatrix} = \begin{bmatrix} F \\ G \end{bmatrix},$$

Assume the system above has a unique solution c .

$$\begin{pmatrix} A & L^T \\ L & -\epsilon I \end{pmatrix} \begin{bmatrix} \mathbf{c}^{(l+1)} \\ \lambda^{(l+1)} \end{bmatrix} = \begin{bmatrix} F \\ G - \epsilon \lambda^{(l)} \end{bmatrix}, \quad (2)$$

where $\lambda^{(0)}$ is a suitable initial guess e.g. $\lambda^{(0)} = 0$,

Assume that $A_s = \frac{1}{2}(A + A^T)$ the symmetric part of A is positive definite with respect to L , i.e., $x^T A_s x \geq 0$ and $x^T A_s x = 0$ with $Lx = 0$ implies $x = 0$. Then, the sequence $(c^{(l+1)})$ defined by the iterative method converges to the solution c for any $\epsilon > 0$.

Furthermore,

$$\|c - c^{(l+1)}\| \leq C\epsilon \|c - c^{(l)}\|$$

for some constant C independent of ϵ and l .

$$Ac^{(l+1)} + L^T \lambda^{(l+1)} = F \quad \text{and (1)}$$

$$Lc^{(l+1)} - \epsilon \lambda^{(l+1)} = G - \epsilon \lambda^{(l)} \quad (2).$$

Multiplying (2) on the left by L^T and substituting $L^T \lambda^{(l+1)}$ into (1) and rewriting (2), we get

$$\left(A + \frac{1}{\epsilon} L^T L\right) c^{(l+1)} = -L^T \lambda^{(l)} + F + \frac{1}{\epsilon} L^T G \quad (3)$$

$$\lambda^{(l+1)} + \frac{1}{\epsilon} Lc^{(l+1)} = \lambda^{(l)} + \frac{1}{\epsilon} G.$$

$A + \frac{1}{\epsilon} L^T L$ is invertible

$$\left(A + \frac{1}{\epsilon} L^T L\right) x = 0 \Rightarrow x = 0.$$

$$0 = x^T \left(A + \frac{1}{\epsilon} L^T L \right) x = x^T \left(A_s + \frac{1}{\epsilon} L^T L \right) x = x^T A_s x + \frac{1}{\epsilon} (Lx)^T (Lx)$$

It follows that $x^T A_s x = 0$ and $(Lx)^T (Lx) = 0$. so $x = 0$.

$c^{(l+1)}$ converges to c

Put $u^{(l+1)} = c^{(l+1)} - c$ and $p^{(l+1)} = \lambda^{(l+1)} - \lambda$

$$\begin{cases} \left(A + \frac{1}{\epsilon} L^T L \right) u^{(l+1)} + L^T p^{(l)} = 0 \\ p^{(l+1)} = p^{(l)} + \frac{1}{\epsilon} L u^{(l+1)}. \end{cases}$$

$$\|p^{(l)}\|^2 - \|p^{(l+1)}\|^2 = \frac{2}{\epsilon} (A_s u^{(l+1)}, u^{(l+1)}) + \frac{1}{\epsilon^2} \|L u^{(l+1)}\|^2.$$

$\{\|p^{(l)}\|\}$ is seen to be decreasing, bounded below by 0 so cvges

Use positive definiteness of $A_s + \frac{1}{\epsilon} L^T L$ to conclude

We prove that $\|c - c^{(l+1)}\| \leq C\epsilon \|c - c^{(l)}\|$

$$\begin{cases} (A + \frac{1}{\epsilon} L^T L) u^{(l+1)} + L^T p^{(l)} = 0 \\ p^{(l+1)} = p^{(l)} + \frac{1}{\epsilon} L u^{(l+1)}, \end{cases}$$

$$A u^{(l+1)} + L^T p^{(l+1)} = 0$$

We write $u^{(l+1)} = \hat{u}^{(l+1)} + \bar{u}^{(l+1)}$ with $\hat{u}^{(l+1)} \in \text{Ker}(L)$ and $\bar{u}^{(l+1)} \in \text{Im}(L^T)$. Note that $L : \text{Im}(L^T) \rightarrow \text{Im}(L)$ has a bounded inverse, so there exists $k_0 > 0$ such that

$$\|\bar{u}^{(l+1)}\| \leq \frac{1}{k_0} \|L u^{(l+1)}\|,$$

from which it follows that

$$\|\bar{u}^{(l+1)}\| \leq \frac{2\epsilon}{k_0} \|p^{(l)}\|$$

To get a bound on $\|\hat{u}^{(l+1)}\|$, we notice that A is invertible on $\text{Ker}(L)$ since $A + \frac{1}{\epsilon}L^T L$ is invertible. This gives for some $\alpha_0 > 0$,

$$\|\hat{u}^{(l+1)}\| \leq \frac{1}{\alpha_0} \sup_{v_0 \in \text{Ker}(L)} \frac{(v_0, A\hat{u}^{(l+1)})}{\|v_0\|} = \sup_{v_0 \in \text{Ker}(L)} \frac{-v_0^T A\bar{u}^{(l+1)}}{\|v_0\|} \leq \|A\| \|\bar{u}^{(l+1)}\|.$$

Putting together, we obtain

$$\|u^{(l+1)}\| \leq C\epsilon \|p^{(l)}\|, \quad \text{for some constant } C > 0$$

To finish, we need a bound on $\|p^{(l)}\|$ in terms of $\|u^{(l)}\|$. It can be shown that one can choose λ_0 such that $p^{(l)} \in \text{Im}(L)$ and since $L^T : \text{Im}(L) \rightarrow \text{Im}(L^T)$ has a bounded inverse,

$$\|p^{(l)}\| \leq \frac{1}{k_0} \|L^T p^{(l)}\|.$$

This completes the proof since $L^T p^{(l)} = -Au^{(l)}$.

Trivariate splines

Let $d \geq 1$ and $r \geq 0$

$$S_d^r(\Omega) = \{p \in C^r(\Omega), p|_t \in P_d, \forall t \in \mathcal{T}\}.$$

$$B_{ijkl}^d(v) = \frac{d!}{i!j!k!l!} b_1^i b_2^j b_3^k b_4^l, \quad i + j + k + l = d.$$

The set $\mathcal{B}^d = \{B_{ijkl}^d(x, y, z), i + j + k + l = d\}$ is a basis for P_d .

$$s|_T = \sum_{i+j+k+l=d} c_{ijkl}^T B_{ijkl}^d,$$

Interpolation

On the tetrahedron $T = \langle v_1, v_2, v_3, v_4 \rangle$ at $\xi_{ijkl} = \frac{iv_1 + jv_2 + kv_3 + lv_4}{d}$

On the edge $\langle v_1, v_2 \rangle$ at $\xi_{ij} = \frac{iv_1 + jv_2}{d}$

$$\sum_{i+j=d} \tilde{c}_{ij} \tilde{B}_{ij}^d(v), \quad \tilde{B}_{ij}^d = \frac{d!}{i!j!} b_1^i b_2^j.$$

$$p = \sum_{i+j+k=d} c_{ijk} B_{ijk}^d, \quad q = \sum_{i+j=d} c_{ij0} B_{ij0}^d$$

$$p = \sum_{i+j+k+l=d} c_{ijkl} B_{ijkl}^d, \quad q = \sum_{i+j+k=d} c_{ijk0} B_{ijk0}^d$$

$$Rc = c_b$$

Derivatives

$D_i c$, $i = 1, 2$ encode respectively the B -net of $\frac{\partial s}{\partial x_i}$.

Integration

$$\int_{\Omega} pq = c^T Id$$

Smoothness conditions

This shows that there's a (I, N) matrix H such that s is in $C^r(\Omega)$ if and only if

$$Hc = 0.$$

Poisson equation

Tetrahedra	d=1	d=2	d=3	d=4
6	6.4017e+00	1.3922e+00	2.3880e-01	3.2070e-02
6*8=48	2.7623e+00	2.9100e-01	2.5136e-02	1.7554e-03
48*8=384	9.5226e-01	4.9066e-02	1.5654e-03	5.0882e-05

Tetrahedra	d=5	d=6
6	4.0221e-03	3.9298e-04
6*8=48	7.9726e-05	4.6256e-06

$$u = \exp(x + y + z)$$

Biharmonic equation

Tetrahedra	6	48
d=2	1.5571e-01	5.9921e-02
d=3	6.4337e-02	9.1870e-03
d=4	1.0803e-02	5.9974e-03
d=5	1.2830e-02	Out of memory

$$u = \exp(-x^2 - y^2 - z^2)$$

Stokes equations

$$u_1 = -\exp(x + 2y + 3z), u_2 = 2 \exp(x + 2y + 3z), u_3 = -\exp(x + 2y + 3z), p = x(1-x)z(1-z)$$

Table 1 Approximation Errors from Trivariate Spline Spaces on \mathcal{I}_1

degrees	u_1	u_2	u_3	p
3	3.3633×10	5.9431×10	4.0397×10	1.3466×10^3
4	1.7010×10	4.4374×10	3.5368×10	3.8562×10^2
5	2.3804	7.3711	5.9629	9.8470×10^1
6	3.9620×10^{-1}	1.2238	1.0311	2.7404×10^1
7	6.7456×10^{-2}	1.9789×10^{-1}	1.6260×10^{-1}	6.8411
Rate	$1.56 \times 10^7 d^{-9.8294}$	$3.22 \times 10^7 d^{-9.6203}$	$2.32 \times 10^7 d^{-9.5463}$	$8.50 \times 10^6 d^{-7.1353}$

Table 2 Approximation Errors from Trivariate Spline Spaces on \mathcal{I}_2

degrees	u_1	u_2	u_3	p
3	1.5083×10	1.8709×10	1.5222×10	4.4382×10^2
4	9.4142×10^{-1}	2.2094	1.8373	3.5278×10^1
5	9.1619×10^{-2}	2.2322×10^{-1}	2.0176×10^{-1}	5.8199
6	8.5128×10^{-3}	2.3520×10^{-2}	1.9276×10^{-2}	7.1884×10^{-1}
Rate	$9.31 \times 10^6 d^{-11.5631}$	$1.24 \times 10^7 d^{-11.1692}$	$1.09 \times 10^7 d^{-11.1901}$	$1.05 \times 10^7 d^{-9.1064}$

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Biharmonic Poisson

$$\epsilon^2 \Delta^2 u - \Delta u = f \text{ in } \Omega$$

$$u = 0, \quad \frac{\partial u}{\partial n} = 0 \text{ in } \partial\Omega$$

$$V = W = H_0^2(\Omega), \quad \epsilon^2 \int_{\Omega} \Delta u \Delta v + \int_{\Omega} \nabla u \cdot \nabla v = \int_{\Omega} f v.$$

$$V_h = \{u \in \mathcal{S}_d^1, u = 0 \text{ and } \partial u / \partial n = 0 \text{ on } \partial\Omega\}$$

$$\equiv \{c \in \mathbf{R}^N, Hc = 0, Rc = 0, Nc = 0\}$$

$$\begin{bmatrix} \epsilon^2 B + K & H^T & R^T & N^T \\ H & 0 & 0 & 0 \\ R & 0 & 0 & 0 \\ N & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} c \\ \lambda_1 \\ \lambda_2 \\ \lambda_3 \end{bmatrix} = \begin{bmatrix} MF \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$|||u|||^2 = \epsilon^2 \sum_{T \in \mathcal{T}_h} \int_T D^2 u : D^2 v \, dx + \sum_{T \in \mathcal{T}_h} \int_T Du : Dv \, dx$$

$$\frac{|||u'_h - u_h|||}{|||u'_h|||}$$

Theorem

For $d \geq 3r + 2$ and $0 \leq m \leq d$ and for $0 \leq k \leq m$.

$$|f - Qf|_{p,k} \leq C|\Delta|^{m+1-k}|f|_{p,m+1}$$

For $u \in H^{d+1}(\Omega)$, for $d \geq 5$,

$$\begin{aligned} \inf_{v \in \mathcal{S}_d^1} \|u - v\|_{\epsilon}^2 &\leq \epsilon^2 h^{2(d-1)} |u|_{d+1}^2 + h^{2d} |u|_{d+1}^2 \\ &= h^{2(d-1)} (h^2 + \epsilon^2) |u|_{d+1}^2 \end{aligned}$$

Using cubic polynomials

ϵ/h	2^{-3}	2^{-4}	2^{-5}	2^{-6}
2^0	$2.1265 \cdot 10^{-1}$	$1.0271 \cdot 10^{-1}$	$1.9464 \cdot 10^{-2}$	$3.3889 \cdot 10^{-3}$
2^{-2}	$1.8980 \cdot 10^{-1}$	$9.4724 \cdot 10^{-2}$	$4.5625 \cdot 10^{-2}$	$8.5138 \cdot 10^{-3}$
2^{-4}	$9.6005 \cdot 10^{-2}$	$4.5406 \cdot 10^{-2}$	$2.2429 \cdot 10^{-2}$	$1.0786 \cdot 10^{-2}$
2^{-6}	$4.0374 \cdot 10^{-2}$	$1.4142 \cdot 10^{-2}$	$6.3389 \cdot 10^{-3}$	$3.0774 \cdot 10^{-3}$
2^{-8}	$3.2036 \cdot 10^{-2}$	$7.5016 \cdot 10^{-3}$	$2.2452 \cdot 10^{-3}$	$8.6681 \cdot 10^{-4}$
2^{-10}	$3.1423 \cdot 10^{-2}$	$6.8540 \cdot 10^{-3}$	$1.6684 \cdot 10^{-3}$	$4.4247 \cdot 10^{-4}$
Poisson (S)	$2.3071 \cdot 10^{-2}$	$5.2191 \cdot 10^{-3}$	$1.2678 \cdot 10^{-3}$	$3.1412 \cdot 10^{-4}$
Poisson	$1.9685 \cdot 10^{-3}$	$2.6203 \cdot 10^{-4}$	$3.3516 \cdot 10^{-5}$	$4.2260 \cdot 10^{-6}$
Biharmonic	$2.1450 \cdot 10^{-1}$	$1.0367 \cdot 10^{-1}$	$1.9654 \cdot 10^{-2}$	$4.3614 \cdot 10^{-3}$

$$u(x, y) = (\sin(\pi x) \sin(\pi y))^2$$

Using polynomials of degree 4

ϵ/h	2^{-3}	2^{-4}	2^{-5}	2^{-6}
2^0	$2.4378 \cdot 10^{-2}$	$5.8674 \cdot 10^{-3}$	$1.0959 \cdot 10^{-3}$	$2.7988 \cdot 10^{-4}$ *
2^{-2}	$2.1598 \cdot 10^{-2}$	$5.1899 \cdot 10^{-3}$	$1.2694 \cdot 10^{-3}$	$2.3759 \cdot 10^{-4}$ *
2^{-4}	$1.0534 \cdot 10^{-2}$	$2.4702 \cdot 10^{-3}$	$6.0576 \cdot 10^{-4}$	$9.4533 \cdot 10^{-5}$
2^{-6}	$4.0397 \cdot 10^{-3}$	$7.6437 \cdot 10^{-4}$	$1.7134 \cdot 10^{-4}$	$4.1365 \cdot 10^{-5}$
2^{-8}	$2.9540 \cdot 10^{-3}$	$3.9977 \cdot 10^{-4}$	$6.1923 \cdot 10^{-5}$	$1.1852 \cdot 10^{-5}$
2^{-10}	$2.8664 \cdot 10^{-3}$	$3.6170 \cdot 10^{-4}$	$4.6177 \cdot 10^{-5}$	$6.2184 \cdot 10^{-6}$
Poisson (S)	$1.9956 \cdot 10^{-3}$	$2.5712 \cdot 10^{-4}$	$3.2459 \cdot 10^{-5}$	$4.5161 \cdot 10^{-6}$
Poisson	$2.4134 \cdot 10^{-4}$	$1.5286 \cdot 10^{-5}$	$9.5869 \cdot 10^{-7}$	$6.2866 \cdot 10^{-10}$
Biharmonic	$2.4605 \cdot 10^{-2}$	$5.9116 \cdot 10^{-3}$	$1.2668 \cdot 10^{-3}$	$3.0533 \cdot 10^{-4}$ *

- 1 Why multivariate splines for PDEs ?
 - Motivation
 - Features of the method
 - Approximation properties
- 2 Refinement of tetrahedra and saddle point problems
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 - Navier-Stokes equations

$$\begin{aligned}\mathbf{u} - \epsilon^2 \Delta \mathbf{u} - \nabla p &= \mathbf{f} \text{ in } \Omega \\ \operatorname{div} \mathbf{u} &= g \text{ in } \Omega \\ \mathbf{u} &= 0 \text{ on } \partial\Omega.\end{aligned}$$

$$W = \{\mathbf{u} \in H_0^1(\Omega)^2, \operatorname{div} \mathbf{u} = g\} \quad V = \{\mathbf{u} \in H_0^1(\Omega)^2, \operatorname{div} \mathbf{u} = 0\}$$

$$\text{Find } \mathbf{u} \in W, \quad \epsilon^2 \int_{\Omega} \nabla \mathbf{u} \cdot \nabla \mathbf{u} + \int_{\Omega} \mathbf{u} \cdot \mathbf{v} = \int_{\Omega} \mathbf{f} \cdot \mathbf{v}, \forall \mathbf{v} \in V.$$

$$W_h = \{\mathbf{c} \in \mathbf{R}^N, H\mathbf{c} = 0, R\mathbf{c} = 0, D\mathbf{c} = G\}$$

$$\begin{bmatrix} \epsilon^2 K + M & H^T & R^T & D^T \\ H & 0 & 0 & 0 \\ R & 0 & 0 & 0 \\ D & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} c \\ \lambda_1 \\ \lambda_2 \\ \lambda_3 \end{bmatrix} = \begin{bmatrix} MF \\ 0 \\ 0 \\ G \end{bmatrix}$$

$$\|\mathbf{v}\|_\epsilon^2 = \|\mathbf{v}\|^2 + \epsilon^2 \|\nabla \mathbf{v}\|^2$$

$$\tilde{V} = \{\mathbf{v} \in L^2(\Omega)^2, \operatorname{div} \mathbf{v} = 0, \mathbf{v} \cdot \mathbf{n} = 0 \text{ on } \partial\Omega\}.$$

Divergence-free vector fields

$$V^d = \{\mathbf{f} \in (H^d(\Omega))^2, \mathbf{f} = \text{curl } \phi, \phi \in H^{d+1}(\Omega)\}$$

and

$$\mathcal{V}_d = \{\mathbf{s} \in (S_d^0(\Omega))^2, \mathbf{s} = \text{curl } S, S \in S_{d+1}^1(\Omega)\}$$

$$\inf_{\mathbf{s} \in \mathcal{V}_d} \|\mathbf{f} - \mathbf{s}\|_2 \leq h^d |\mathbf{f}|_d \text{ and } \inf_{\mathbf{s} \in \mathcal{V}_d} \|\nabla \mathbf{f} - \nabla \mathbf{s}\|_2 \leq h^{d-1} |\mathbf{f}|_d,$$

$$\inf_{\substack{\mathbf{s} \in \mathcal{V}_d \\ \mathbf{s}=0 \text{ on } \partial\Omega}} \|\mathbf{u} - \mathbf{s}\|_\epsilon \leq (\epsilon h^{d-1} + h^d) |\mathbf{u}|_d, d \geq 4.$$

Using cubic polynomials

ϵ/h	2^{-2}	2^{-3}	2^{-4}	2^{-5}
2^{-0}	$1.0061 \cdot 10^{-1}$	$2.3718 \cdot 10^{-2}$	$5.8212 \cdot 10^{-3}$	$1.4467 \cdot 10^{-3}$
2^{-2}	$8.9696 \cdot 10^{-2}$	$2.1014 \cdot 10^{-2}$	$5.1490 \cdot 10^{-3}$	$1.2791 \cdot 10^{-3}$
2^{-4}	$4.6793 \cdot 10^{-2}$	$1.0240 \cdot 10^{-2}$	$2.4505 \cdot 10^{-3}$	$6.0662 \cdot 10^{-4}$
2^{-6}	$2.3921 \cdot 10^{-2}$	$3.8922 \cdot 10^{-3}$	$7.5706 \cdot 10^{-4}$	$1.8105 \cdot 10^{-4}$
2^{-8}	$2.0934 \cdot 10^{-2}$	$2.8315 \cdot 10^{-3}$	$3.9537 \cdot 10^{-4}$	$1.3380 \cdot 10^{-4}$
0.00	$2.0708 \cdot 10^{-2}$	$2.7403 \cdot 10^{-3}$	$3.5598 \cdot 10^{-4}$	$5.8378 \cdot 10^{-5*}$
Darcy	$4.3368 \cdot 10^{-3}$	$8.6596 \cdot 10^{-4}$	$1.7278 \cdot 10^{-5*}$	$1.9652 \cdot 10^{-2+}$

$$\mathbf{u} = \text{curl}(\sin(\pi x)^2 \sin(\pi y)^2), \quad \text{and } p = \sin(\pi x)$$

Why multivariate splines for PDEs ?

Refinement of tetrahedra and saddle point problems

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Robustness of the method for singular perturbation problems

Nonlinear problems

Fourth order singular problem

Darcy-Stokes equation

Pressure P_3/P_4

ϵ/h	2^{-2}	2^{-3}	2^{-4}
2^{-0}	7.9689	3.1380	$8.9906 \cdot 10^{-1}$
2^{-2}	$5.0369 \cdot 10^{-1}$	$1.9471 \cdot 10^{-1}$	$5.6019 \cdot 10^{-2}$
2^{-4}	$3.3121 \cdot 10^{-2}$	$1.1486 \cdot 10^{-2}$	$3.4025 \cdot 10^{-3}$
2^{-6}	$2.9191 \cdot 10^{-3}$	$6.8701 \cdot 10^{-4}$	$1.9454 \cdot 10^{-4}$
2^{-8}	$1.5185 \cdot 10^{-3}$	$9.0396 \cdot 10^{-5}$	$1.1731 \cdot 10^{-5}$
0.00	$1.4690 \cdot 10^{-3}$	$6.9076 \cdot 10^{-5}$	$2.7063 \cdot 10^{-6}$
Darcy	$1.4690 \cdot 10^{-3}$	$6.9076 \cdot 10^{-5}$	$2.7063 \cdot 10^{-6}$

Using polynomials of degree 4

ϵ/h	2^{-2}	2^{-3}	2^{-4}	2^{-5}
2^{-0}	$1.7486 \cdot 10^{-2}$	$1.0465 \cdot 10^{-3}$	$6.0180 \cdot 10^{-5}$	$4.0017 \cdot 10^{-6}*$
2^{-2}	$1.5541 \cdot 10^{-2}$	$9.2597 \cdot 10^{-4}$	$5.3232 \cdot 10^{-5}$	$1.7016 \cdot 10^{-5}+$
2^{-4}	$7.8996 \cdot 10^{-3}$	$4.4325 \cdot 10^{-4}$	$2.5327 \cdot 10^{-5}$	$1.5584 \cdot 10^{-4}+$
2^{-6}	$3.6643 \cdot 10^{-3}$	$1.4529 \cdot 10^{-4}*$	$2.0257 \cdot 10^{-5}$	$1.2314 \cdot 10^{-4}+$
2^{-8}	$3.0271 \cdot 10^{-3}$	$8.5252 \cdot 10^{-5}$	$8.3301 \cdot 10^{-6}*$	$1.0183 \cdot 10^{-2}+$
0.00	$2.9751 \cdot 10^{-3}$	$7.8673 \cdot 10^{-5}*$	$1.7288 \cdot 10^{-3}+$	$1.7788 \cdot 10^{-2}+$
Darcy	$6.4139 \cdot 10^{-4}$	$2.1203 \cdot 10^{-5}*$	$1.2037 \cdot 10^{-4}+$	$1.5822 \cdot 10^{-2}+$

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Robustness of the method for singular perturbation problems

Nonlinear problems

Fourth order singular problem

Darcy-Stokes equation

Pressure P_4/P_5

ϵ/h	2^{-2}	2^{-3}
2^{-0}	7.5061	$9.0088 \cdot 10^{-1}$
2^{-2}	$4.7198 \cdot 10^{-1}$	$5.6434 \cdot 10^{-2}$
2^{-4}	$2.9984 \cdot 10^{-2}$	$3.5811 \cdot 10^{-3}$
2^{-6}	$2.5980 \cdot 10^{-3}$	$2.2229 \cdot 10^{-4}$
2^{-8}	$1.1138 \cdot 10^{-3}$	$1.7635 \cdot 10^{-5}$
0.00	$1.0343 \cdot 10^{-3}$	$9.3820 \cdot 10^{-8}$
Darcy	$1.0343 \cdot 10^{-3}$	$9.3820 \cdot 10^{-8}$

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$$\begin{cases} -\nu \Delta \mathbf{u} + \sum_{j=1}^3 u_j \frac{\partial \mathbf{u}}{\partial x_j} + \nabla p = \mathbf{f} & \text{in } \Omega, \\ \operatorname{div} \mathbf{u} = 0 & \text{in } \Omega. \end{cases}$$

$$V_0 = \{\mathbf{v} \in H_0^1(\Omega)^3, \operatorname{div} \mathbf{v} = 0\}$$

$$L_0^2(\Omega) = \{u \in L^2(\Omega), \int_{\Omega} u = 0\} \quad \text{and}$$

$$H^{\frac{1}{2}}(\partial\Omega) = \{\tau(u), u \in H^1(\Omega)\},$$

Weak formulation : Find $\mathbf{u} \in H^1(\Omega)^3$ such that

$$\nu \int_{\Omega} \nabla \mathbf{u} \cdot \nabla \mathbf{v} + \sum_{j=1}^3 \int_{\Omega} u_j \frac{\partial \mathbf{u}}{\partial x_j} \cdot \mathbf{v} = \int_{\Omega} \mathbf{f} \cdot \mathbf{v} \quad \forall \mathbf{v} \in V_0$$

$$\operatorname{div} \mathbf{u} = 0 \quad \text{in } \Omega$$

$$\mathbf{u} = \mathbf{g} \quad \text{on } \partial\Omega$$

$$\nu \bar{K} \mathbf{c} + \bar{B}(\mathbf{c}) \mathbf{c} + L^T \lambda = \bar{M} \mathbf{F}$$

$$L \mathbf{c} = \bar{G}$$

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Robustness of the method for singular perturbation problems

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Navier-Stokes equations

Lid driven Cavity Flow

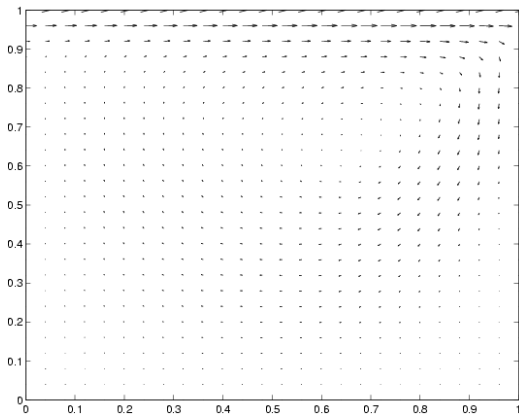


FIG.: 3D fluid profile in the $x - y$ plane

Why multivariate splines for PDEs ?

Refinement of tetrahedra and saddle point problems

Linear problems

Robustness of the method for singular perturbation problems

Nonlinear problems

Navier-Stokes equations

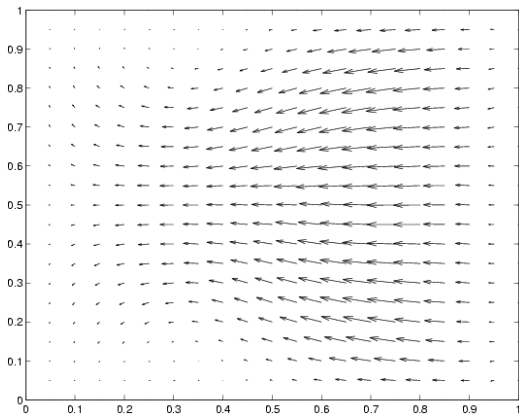


FIG.: 3D fluid profile in the $y - z$ plane

Why multivariate splines for PDEs ?

Refinement of tetrahedra and saddle point problems

Linear problems

Robustness of the method for singular perturbation problems

Nonlinear problems

Navier-Stokes equations

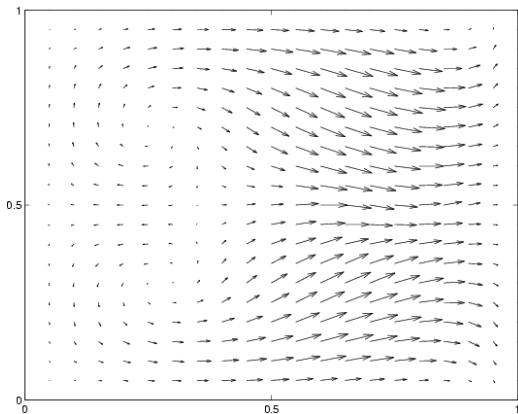


FIG.: 3D fluid profile in the $x - z$ plane

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