Finite Elements for Symmetric Tensors

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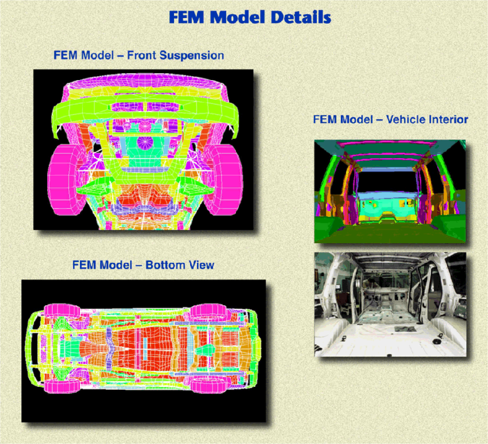
Outline

Applications of finite element



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Linear Elasticity Problem

Displacement $u_i = x_i' - x_i$

$$dl'^2 = dl^2 + \sum_{i,k} 2(\epsilon u)_{ik}\, dx_i\, dx_k$$

$$(\epsilon u)_{ik} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_k} + \frac{\partial u_k}{\partial x_i} + \sum_l \frac{\partial u_l}{\partial x_k}\frac{\partial u_l}{\partial x_i}\right)$$

For small deformations, the strain tensor is

$$(\epsilon u)_{ik} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_k} + \frac{\partial u_k}{\partial x_i}\right)$$

$\sum_k \sigma_{ik} n_k$ is the ith component of the force acting on the element of surface ds with normal $\mathbf{n}$.

Linear Elasticity Problem

$$A\,\sigma = \epsilon(u) \quad \text{in } \Omega$$
$$\text{div}\,\sigma = f \quad \text{in } \Omega$$
$$u = 0 \quad \text{on } \partial\Omega,$$

- $u : \Omega \longrightarrow \mathbb{R}^n$ measures the displacement
- $\sigma : \Omega \longrightarrow \mathbb{S}$ measures the internal forces where Sym is the space of $n \times n$ symmetric matrices.

$$\epsilon(u) = \frac{1}{2}(\nabla u + (\nabla u)^T)$$

A is called compliance tensor and f encodes the body forces.

Variational Formulations

1. Primal variational principle : displacement over $v = 0$ on $\partial\Omega$

$$\int_\Omega \frac{1}{2} A^{-1}\epsilon(v) : \epsilon(v) + f \cdot v$$

2. Dual Variational Principle : stress field over $\operatorname{div} \tau = f$

$$\int_\Omega \frac{1}{2} A\tau : \tau$$

3. Mixed variational principle : stress field and displacement

$$\int_\Omega \left(\frac{1}{2} A\tau : \tau + \operatorname{div} \tau \cdot v - f \cdot v \right) dx$$

Mixed Weak Formulation

Find $\sigma \in \Sigma = H(\mathrm{div}, \Omega, \mathbb{S}) = \{\sigma \in L^2(\Omega, \mathbb{S}), \mathrm{div}\,\sigma \in L^2(\Omega, \mathbb{R}^n)\}$ and $u \in V = L^2(\Omega, \mathbb{R}^n)$ such that

$$
\begin{aligned}
(A\sigma, \tau) + (\mathrm{div}\,\tau, u) &= 0, \quad \forall \tau \in \Sigma, \\
(\mathrm{div}\,\sigma, v) &= (f, v), \quad \forall v \in V.
\end{aligned}
$$

Discrete problem posed on $\Sigma_h \subset \Sigma$ and $V_h \subset V$

$$
\sigma = (\sigma_{ij})_{i,j=1,\dots,n} \quad \sigma_{ij} = \sigma_{ji} \quad u = (u_i)_{i=1,\dots,n}
$$

Stable approximations ?

Mixed Weak Formulation

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Stable approximations ?

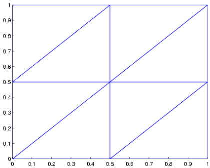
Finite element spaces

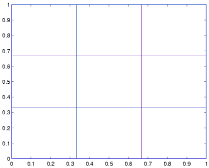
- K is a closed subset of $\mathbb{R}^n$ with a nonempty interior and a Lipshitz continuous boundary
- P_K is a finite dimensional space of vector valued or matrix valued functions defined over the set K
- Θ_K is a finite set of linearly independent linear functionals, $\theta_i, i = 1, \ldots, N$ referred to as degrees of freedom of the finite element, defined over the set P_K.

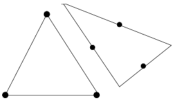
It is assumed that the set Θ_K is P_K-unisolvent in the sense that

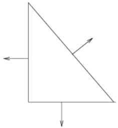
$$\theta_i(p) = 0, i = 1, \ldots, N \Longrightarrow p \equiv 0$$

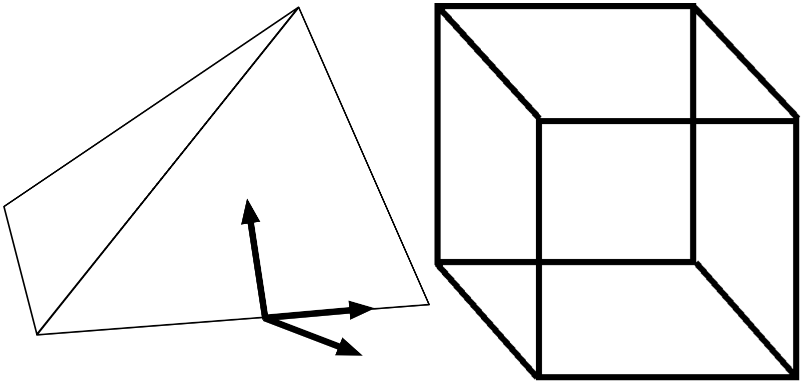
Examples











Continuity conditions for conformity

A piecewise infinitely differentiable function belongs to H^k if and only if it is C^{k-1}
A piecewise smooth vector field v is in $H(\text{div})$ if and only if for each common face $F = K_1 \cap K_2$ of the triangulation, the trace on F of the normal component $n \cdot v|_{K_1}$ and $n \cdot v|_{K_2}$ is the same

$$\sigma = \begin{pmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{12} & \sigma_{22} & \sigma_{23} \\ \sigma_{13} & \sigma_{23} & \sigma_{33} \end{pmatrix} \qquad \sigma \text{ symmetric} \quad u = \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix}$$

Stable approximations ?

Surprisingly stable mixed finite elements for elasticity have been difficult to construct.

Composite Elements Fraejis de Veubeke (1965), Watwood and Hartz (1968), Johnson and Mercier (1978), Arnold-Douglas-Gupta (1984)

Modified variational problems Arnold and Falk(1988), Becache, Joly and Tsogka (2002), . . .

Weakly imposed symmetry condition Arnold, Brezzi and Douglas (1984), Stenberg (1986), Morley (1989), Arnold, Falk and Winther (2006) and (2007), . . .

Using polynomial shape functions D. Arnold and R. Winther, (2002), D. Arnold and R. Winther, (2003), S. Adams and B. Cockburm, (2004), Yi (2005) and (2006), Hu and Shi, (2007), Man, Hu and Shi, (2008).

My own contributions : D. Arnold and G. Awanou, Rectangular Mixed Finite Elements for Elasticity. Math. Models and Methods in Appl. Sci. (2005).

G. Awanou A Rotated Nonconforming Rectangular Mixed Finite Element for Elasticity. To appear in Calcolo, (2009).

D. Arnold, G. Awanou and R. Winther, Finite elements for symmetric tensors in three dimensions. Mathematics of Computation, (2008).

D. Arnold and G. Awanou, Rectangular Mixed Elements for Elasticity with weakly imposed symmetry condition, (In preparation).

D. Arnold, G. Awanou and R. Winther, Nonconforming Tetrahedral Mixed Finite Elements for Elasticity, (In preparation).

Abstract Mixed Formulation

Find $\sigma \in \Sigma = H(\mathrm{div}, \Omega, \mathbb{S})$ and $u \in V = L^2(\Omega, \mathbb{R}^n)$ such that

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Discrete Problem

$$\begin{aligned} a(\sigma_h, \tau_h) + b(\tau_h, u_h) &= 0, \quad \forall \tau_h \in \Sigma_h, \quad \Sigma_h \subset \Sigma \\ b(\sigma_h, v_h) &= (f, v_h), \quad \forall v_h \in V_h, V_h \subset V. \end{aligned}$$

Brezzi's stability conditions

$\Sigma_h \subset \Sigma$ and $V_h \subset V$

Sufficient conditions for optimal error bounds

First Brezzi condition $\exists \alpha > 0$ independent of h such that

$$a(\tau, \tau) \geq \alpha \|\tau\|_\Sigma^2$$

for all τ in K_h where

$$K_h = \{\tau \in \Sigma_h : b(\tau, v) = 0, \forall v \in V_h\}$$

Second Brezzi condition $\exists \beta > 0$ independent of h such that

$$\sup_{\tau \in \Sigma_h} \frac{b(\tau, v)}{\|\tau\|_\Sigma} \geq \beta \|v\|_V \quad \forall v \in V_h$$

$$\|\sigma - \sigma_h\|_\Sigma + \|u - u_h\|_V \leq \gamma \{\inf_{\tau \in \Sigma_h} \|\sigma - \tau\|_\Sigma + \inf_{v_h \in V_h} \|u - v_h\|_V\}$$

with γ independent of h.

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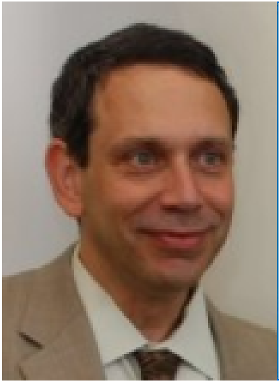
Commutative Diagram

Sufficient conditions for stability

- $\operatorname{div} \Sigma_h \subset V_h$.
- There exists a linear operator $\Pi_h : H^1(\Omega, \mathbb{S}) \to \Sigma_h$, bounded in $\mathcal{L}(H^1, L^2)$ uniformly with respect to h, and such that with $P_h : L^2(\Omega, \mathbb{R}^n) \to V_h$ denoting the L^2-projection

$$\begin{array}{ccc} H(\mathrm{div}, \Omega, \mathbb{S}) & \xrightarrow{\mathrm{div}} & L^2(\Omega, \mathbb{R}^n) \\ \downarrow \Pi_h & & \downarrow P_h \\ \Sigma_h & \xrightarrow{\mathrm{div}} & V_h \end{array}$$




Arnold-Winther 2D elements

Elasticity Differential Complex

$$\begin{array}{ccccccccc}
\mathcal{P}_1(\Omega) & \xrightarrow{\subseteq} & H^2(\Omega) & \xrightarrow{J} & H(\mathrm{div},\Omega,\mathbb{S}) & \xrightarrow{\mathrm{div}} & L^2(\Omega,\mathbb{R}^2) & \rightarrow 0 \\
 & & \downarrow I_h & & \downarrow \Pi_h & & \downarrow P_h & \\
\mathcal{P}_1(\Omega) & \xrightarrow{\subseteq} & Q_h & \xrightarrow{J} & \Sigma_h & \xrightarrow{\mathrm{div}} & V_h & \rightarrow 0
\end{array}$$

$$Jq := \begin{pmatrix} \frac{\partial^2 q}{\partial y^2} & -\frac{\partial^2 q}{\partial x \partial y} \\ -\frac{\partial^2 q}{\partial x \partial y} & \frac{\partial^2 q}{\partial x^2} \end{pmatrix}$$

$$\mathcal{P}_1(T) \xrightarrow{\subseteq} \mathcal{P}_5(T) \xrightarrow{J} \mathcal{P}_3(T,\mathbb{S}) \xrightarrow{\mathrm{div}} \mathcal{P}_2(T,\mathbb{R}^2) \rightarrow 0$$

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But which degrees of freedom ? dim $Q_h = 6V + E$

Condition for σ to be in $H(\text{div})$ is that σn is continuous
From the diagram

$$\mathcal{P}_1(T) \xrightarrow{\subset} \mathcal{P}_5(T) \xrightarrow{J} \mathcal{P}_3(T, \mathbb{S}) \xrightarrow{\text{div}} \mathcal{P}_2(T, \mathbb{R}^2) \to 0$$

possible choices are $\Sigma_T = \mathcal{P}_3(T, \mathbb{S})$ and $V_T = \mathcal{P}_2(T, \mathbb{R}^2)$

$$3(V - E + T) - (6V + E) + (xV + yE + zT) - wT = 0$$

which gives $x = 3, y = 4$ and $z = w - 3$.
Degrees of freedom

1. the values of each component of $\tau(x)$ at the vertices of T (9 degrees of freedom)
2. the first two moments of each component of τn on each edge (12 degrees of freedom)

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But recall that for stability, need $\operatorname{div} \Sigma_h \subset V_h$ and commutativity property $P_h \operatorname{div} \tau = \operatorname{div} \Pi_h \tau, \tau \in H(\operatorname{div}, \Omega, \mathbb{S})$ or

$$\int_T \mathrm{div}\,(\tau - \Pi_h \tau) \cdot v\, dx = -\int_T (\tau - \Pi_h \tau) : \epsilon(v)\, dx + \int_{\partial T} (\tau - \Pi_h \tau) n \cdot v\, ds$$

Sufficient conditions are $v \in \operatorname{span}\{1, x\}$ on each edge

$$\Sigma_T = \{\ \tau \in \mathcal{P}_3(T, \mathbb{S}),\ \mathrm{div}\ \tau \in V_T \}\ \text{and}\ V_T = \mathcal{P}_1(T, \mathbb{R}^2)$$

So $w = 6$ and $z = 3$.

3 the values of $\int_T \tau : \phi$ for all ϕ in $\epsilon(V_T)$

3D elasticity sequence

$$\mathcal{T} \to H^1(\Omega,\mathbb{R})^3 \xrightarrow{\epsilon} H(\mathrm{curl\ curl}^*,\Omega,\mathbb{S}) \xrightarrow{\mathrm{curl\ curl}^*} H(\mathrm{div},\Omega,\mathbb{S}) \xrightarrow{\mathrm{div}} L^2(\Omega,\mathbb{R}^3) \to 0$$

$$\mathcal{T} \xrightarrow{\subset} \mathcal{P}_{k+4}(\Omega,\mathbb{R}^3) \xrightarrow{\epsilon} \mathcal{P}_{k+3}(\Omega,\mathbb{S}) \xrightarrow{\mathrm{curl\ curl}^*} \mathcal{P}_{k+1}(\Omega,\mathbb{S}) \xrightarrow{\mathrm{div}} \mathcal{P}_k(\Omega,\mathbb{R}^3) \to 0.$$

$$\mathcal{T} \to R_h \xrightarrow{\epsilon} Q_h \xrightarrow{\mathrm{curl\ curl}^*} \Sigma_h \xrightarrow{\mathrm{div}} V_h \to 0$$

But both Q_h and Σ_h are spaces of symmetric matrix fields

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Features of the 2D elements

V_K space of discontinuous piecewise polynomials, $\mathcal{P}_s(K;\mathbb{R}^2)$

$$\Sigma_K = \{\,\tau \in \mathcal{P}_k(K;\mathbb{S}) \mid \operatorname{div}\tau \in V_K\,\}$$

Σ_K space of matrix fields with degrees of freedom

- vertex degrees of freedom
- degrees of freedom for τn
- $\int_K \tau : \epsilon(v), v \in V_K$
- $\int_K \tau : \phi,\ \phi$ in

$$\{\tau \in \Sigma_K,\ \text{div}\ \tau = 0, \tau n = 0 \text{ on } \partial K\}$$

k=s+2, $s \geq 1$

Discrete stress field

Postulate $V_K = \mathcal{P}_s(K;\mathbb{R}^3)$, and

$$\Sigma_K = \{\, T \in \mathcal{P}_k(K;\mathbb{S}) \mid \operatorname{div} T \in V_K \,\}$$

$\dim \mathcal{P}_k(T,\mathbb{R}) = \frac{(k+2)(k+1)}{2}$
Need $6 \times 4 = 24$ vertex degrees of freedom
edge with normal n_- and n_+, $n_-' T n_+ \in \mathcal{P}_k(e,\mathbb{R})$, $k-1$ d.o.f.
Same for $n_-' T n_-$, $n_-' T t$, $n_+' T n_+$, $n_+' T t$
$\frac{(k+2)(k+1)}{2} - 3 - 3(k-1) = \frac{(k-1)(k-2)}{2} = \dim \mathcal{P}_{k-3}(f,\mathbb{R})$
Face degrees of freedom $\int_f Tn \cdot v\, dx$, $v \in \mathcal{P}_{k-3}(f,\mathbb{R}^3)$
$\int_K Tn \cdot v\, dx = 0$, $v \in V_K$ Need $s \leq k-3$

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Low order elements

$$\Sigma_K = \{\, T \in \mathcal{P}_4(K;\mathbb{S}) \mid \operatorname{div} T \in \mathcal{P}_1(K;\mathbb{R}^3) \,\} \quad \text{and } V_K = \mathcal{P}_1(K;\mathbb{R}^3)$$

$\dim \Sigma_K \leq 162$ $\qquad$ $\dim V_K = 12$

Degrees of freedom

1. vertex d.o.f . ($4 \times 6 = 24$)
2. edge d.o.f. ($6 \times 3 \times 5 = 90$)
3. face d.o.f. ($4 \times 3 \times 3 = 36$)
4. $\int_K T : U, U \in \epsilon(V_K)$ (6 d.o.f.)
5. the value of the moments $\int_K T : U \; dx$, $U \in M_4(K)$, (6 d.o.f.)

$$M_4(K) := \{\, T \in \mathcal{P}_4(K;\mathbb{S}) \mid \operatorname{div} T = 0 \text{ and } Tn = 0 \text{ on } \partial K \,\}.$$

Somewhat similar space, Adams-Cockburn

Low order elements

$$\Sigma_K = \{\, T \in \mathcal{P}_4(K;\mathbb{S}) \mid \operatorname{div} T \in \mathcal{P}_1(K;\mathbb{R}^3) \,\} \quad \text{and } V_K = \mathcal{P}_1(K;\mathbb{R}^3)$$

$\dim \Sigma_K \leq 162$ $\qquad$ $\dim V_K = 12$

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Somewhat similar space, Adams-Cockburn

Higher order elements

$$\Sigma_K = \{\, T \in \mathcal{P}_{k+3}(K;\mathbb{S}) \mid \operatorname{div} T \in \mathcal{P}_k(K;\mathbb{R}^3) \,\} \quad \text{and } V_K = \mathcal{P}_k(K;\mathbb{R}^3)$$

$$||S - S_h||_0 \le C\, h^{k+2} ||S||_{k+2}$$
$$||u - u_h||_0 \le C\, h^{k+1} ||u||_{k+1}$$

Reduced space is O(h^3) for stress and O(h^2) for displacement

$$\begin{array}{ccccccccc}
\mathcal{T} \to C^\infty(\Omega;\mathbb{R}^3) & \xrightarrow{\epsilon} & C^\infty(\Omega;\mathbb{S}) & \xrightarrow{L} & C^\infty(\Omega;\mathbb{S}) & \xrightarrow{\operatorname{div}} & C^\infty(\Omega;\mathbb{R}^3) & \to 0 \\
\downarrow \Pi_h^R & & \downarrow \Pi_h^Q & & \downarrow \pi_h^\Sigma & & \downarrow P_h & \\
\mathcal{T} \to R_h & \xrightarrow{\epsilon} & Q_h & \xrightarrow{L} & \Sigma_h & \xrightarrow{\operatorname{div}} & V_h & \to 0
\end{array}$$

Higher order elements

$$\Sigma_K = \{\, T \in \mathcal{P}_{k+3}(K;\mathbb{S}) \mid \operatorname{div} T \in \mathcal{P}_k(K;\mathbb{R}^3)\,\} \quad \text{and } V_K = \mathcal{P}_k(K;\mathbb{R}^3)$$

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 & \downarrow \Pi_h^R & & \downarrow \Pi_h^Q & & \downarrow \pi_h^\Sigma & & \downarrow P_h & \\
\mathcal{T} & \rightarrow \quad R_h & \xrightarrow{\epsilon} & Q_h & \xrightarrow{L} & \Sigma_h & \xrightarrow{\operatorname{div}} & V_h & \rightarrow 0
\end{array}$$

The divergence of the stress field is no longer L^2 integrable

Consistency error has to be shown to be bounded

No vertex degrees of freedom and much less d.o.f.

Triangular Arnold-Winther 02 (12-3)
2D Rectangular Awanou 09 (16-3), Yi 05 (19-6), Yi 06 (13-4), Hu-Shi 07 (12-4)
Tetrahedral Arnold-Awanou-Winther 09 (36-6)
3D Rectangular Yi 05 (60-12), Man-Hu-Shi 08 (54-12)

O(h) for stress and displacement (except Yi)

$\mathbb{M}$ denotes the space of 2×2 matrix fields
$\text{as}(\tau) = \tau_{21} - \tau_{12}$.
Find $(\sigma, u, \gamma) \in \Sigma \times V \times Q \subset H(\text{div}, \Omega, \mathbb{M}) \times L^2(\Omega, \mathbb{R}^2) \times L^2(\Omega, \mathbb{R})$ such that

$$\begin{aligned} (A\sigma, \tau) + (\text{div}\,\tau, u) + (\text{as}\,\tau, \gamma) &= 0, \quad \tau \in H(\text{div}, \Omega, \mathbb{M}), \\ (\text{div}\,\sigma, v) &= (f, v), \quad v \in L^2(\Omega, \mathbb{R}^2), \\ (\text{as}\,\sigma, q) &= 0, \quad q \in L^2(\Omega, \mathbb{R}). \end{aligned}$$

Discrete problem posed on $\Sigma_h \subset \Sigma$, $V_h \subset V$ and $Q_h \subset V$.
Allows piecewise constant functions for stress and rotation on triangular, tetrahedral and rectangular in two and three dimensions.

Advantages and Disadvantages

Conforming elements with symmetric stress fields

1. Vertex degrees of freedom
2. High number of degrees of freedom
3. But conformity useful in some situations

Nonconforming elements with symmetric stress fields

1. No vertex degrees of freedom and low number of d.o.f.
2. Can potentially be extended to arbitrary quadrilaterals

Elements with weakly imposed symmetry conditions

1. Typically simpler than conforming elements
2. But comparable to nonconforming elements

Fully nonconforming elements ?