Proposed Course: Math 760: Finite element methods for Partial Differential Equations

Mathematical modeling of physical phenomena has become the standard tool for the investigation of numerous problems in science and engineering but often the resulting equations do not have solutions that can be represented by simple mathematical formulas. Hence the development and the analysis of numerical methods. Math 666 introduces finite difference methods for solving partial differential equations and programming the finite element method. The present course introduces the mathematical theory of finite elements and new paradigms in computation, i.e. the spline element method and differential complexes tools for mixed finite elements.

In finite element methods the primary challenge is to create an equation that approximates the equation to be studied, but is numerically stable, meaning that errors in the input data and intermediate calculations do not accumulate and cause the resulting output to be meaningless. Recent years have seen tremendous progress in the transfer of basic ideas and concepts from differential geometry, algebraic topology and homological algebra in the numerical solution of partial differential equations, when it was realized that stability can be achieved by mimicking geometric, algebraic and homological structures in the partial differential equations. This point of view has led to the solution of previously open problems on mixed finite element methods.

The spline element method uses piecewise polynomials of arbitrary degree and Lagrange multipliers to enforce continuity and smoothness conditions as well as other type of constraints. However, unlike other methods that also use Lagrange multipliers, the constraints here are enforced exactly. It leads to flexible, robust, efficient and accurate approximations allowing easy implementation, the flexibility of using polynomials of different degrees on different elements and the simplicity of a posteriori error estimates since the method is conforming.

This course provides interdisciplinary training for graduate students. It will be self-contained.

Prerequisities are Math 662 or Math 666 or equivalent higher level classes or consent of the instructor. Grades will be based on homeworks and group projects. It is good practice when taking a topic course to try your hands at a research problem. The projects to choose from are

- Mixed finite elements in topology optimization.
- Mixed finite elements for the simulation of the deformation of plastic materials.
- Differential complexes for composite elements for linear elasticity.
- Homology of smooth splines
- Spline element method for the bi-wave equation.
- Spline element method for Cahn-Hilliard equations
- Multigrid methods for the spline element method.

References

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