Differential Complexes and Mixed Finite Elements for Elasticity

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Rectangular mixed finite elements for plane elasticity

Overview

Elasticity Equations: strain, stress and displacement

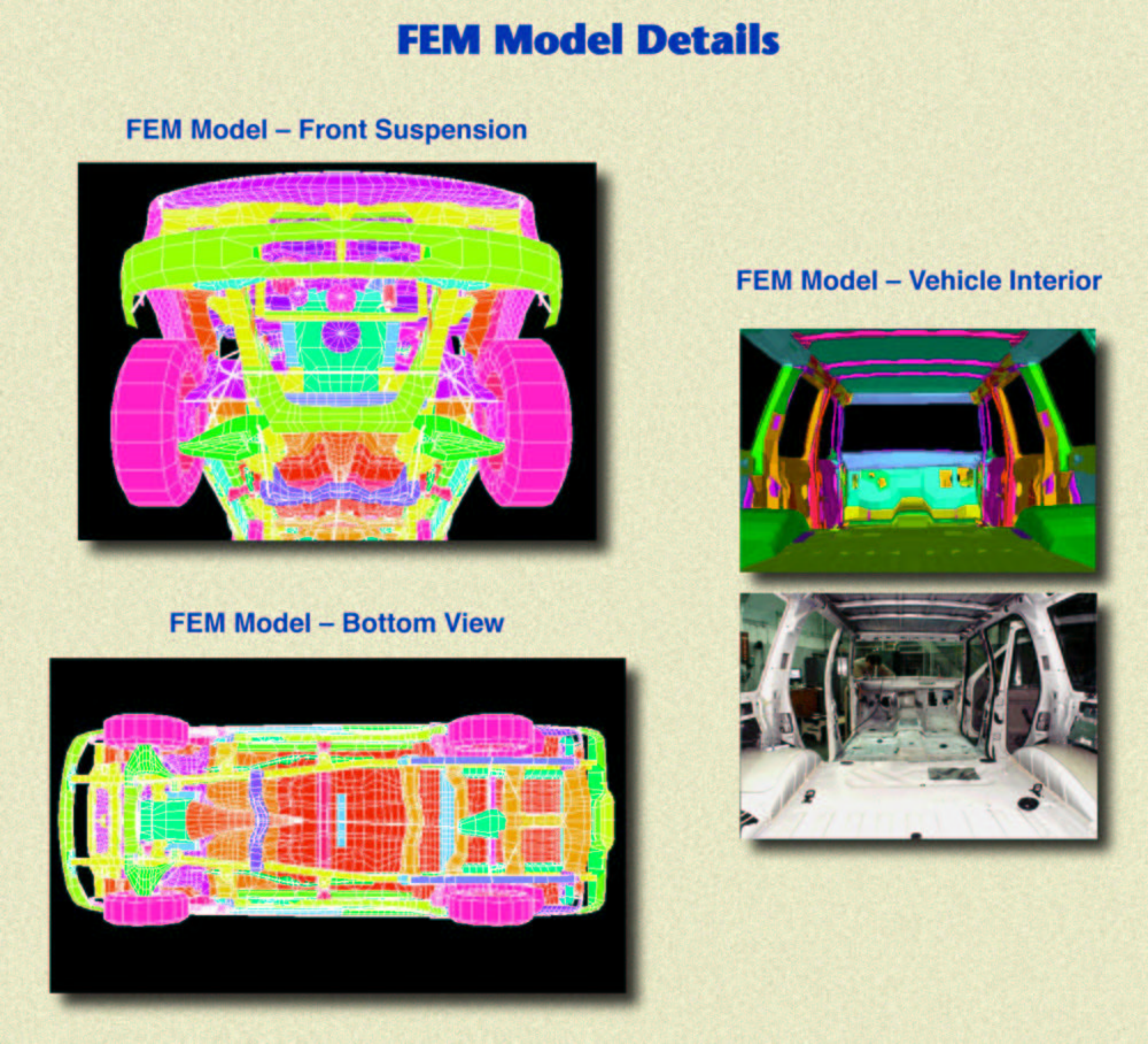
Variational Formulations

Mixed Finite Elements and Stability

Elasticity Differential Complex

Rectangular Mixed Elements

Applications



FEM Model Details

FEM Model – Front Suspension

FEM Model – Vehicle Interior

FEM Model – Bottom View

http://www.epm.ornl.gov/SC98/car.html

The Elasticity Problem

Displacement $u_i = x'_i - x_i$

$$dl'^2 = dl^2 + \sum_{i,k} 2(\epsilon u)_{ik}\, dx_i\, dx_k$$

$$(\epsilon u)_{ik} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_k} + \frac{\partial u_k}{\partial x_i} + \sum_l \frac{\partial u_l}{\partial x_k}\frac{\partial u_l}{\partial x_i}\right)$$

For small deformations, the strain tensor is

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The stress tensor is a second order tensor which measures internal forces

$\sum_k \sigma_{ik} n_k$ is the ith component of the force acting on the element of surface ds with normal $\mathbf{n}$

Equilibrium condition $\operatorname{div} \sigma = f$

Balance of angular momentum $\sigma = \sigma^T$

Linear relationship between stress and strain (Hooke's law) for isotropic material $A\sigma = \epsilon(u)$

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Elasticity Problem Given $f \in L^2(\Omega, \mathbb{R}^2)$, find $u \in L^2(\Omega, \mathbb{R}^2)$ and σ in

$H(\operatorname{div}, \Omega, \mathrm{Sym}) := \{\sigma \in L^2(\Omega, \mathbb{R}^{2\times 2}), \sigma_{ik} = \sigma_{ki}, \operatorname{div} \sigma \in L^2(\Omega, \mathbb{R}^2)\}$

such that

$$\begin{aligned} A\,\sigma &= \epsilon(u) \quad \text{in } \Omega \\ \operatorname{div} \sigma &= f \quad \text{in } \Omega \\ u &= 0 \quad \text{on } \partial\Omega \end{aligned}$$

Variational Formulations

1. Primal variational principle: displacement over $v = 0$ on $\partial\Omega$

$$\int_\Omega \frac{1}{2} A^{-1} \epsilon(v) : \epsilon(v) + f \cdot v$$

2. Dual Variational Principle: stress field over $\operatorname{div} \tau = f$

$$\int_\Omega \frac{1}{2} A\tau : \tau$$

3. Mixed variational principle: stress field and displacement

$$\int_\Omega \left(\frac{1}{2} A\tau : \tau + \operatorname{div} \tau \cdot v - f \cdot v \right) dx$$

Abstract Weak Formulation

Find $\sigma \in \Sigma = H(\text{div}, \Omega, \text{Sym})$ and $u \in V = L^2(\Omega, \mathbb{R}^2)$ such that

$$\int_\Omega A\sigma : \tau + \text{ div } \tau \cdot u + \text{ div } \sigma \cdot v = \int_\Omega f \cdot v$$

for all $\tau \in \Sigma$ and $v \in V$.

$$a(\sigma, \tau) + b(\tau, u) + b(\sigma, v) = F(v)$$

$$a(\sigma, \tau) = \int_\Omega A\sigma : \tau \qquad b(\sigma, u) = \int_\Omega \text{div } \sigma \cdot u \quad F(v) = \int_\Omega f \cdot v$$

$$\begin{cases} a(\sigma, \tau) + b(\tau, u) & = 0 \quad \forall \tau \in \Sigma \\ b(\sigma, v) & = (f, v) \quad \forall v \in V \end{cases}$$

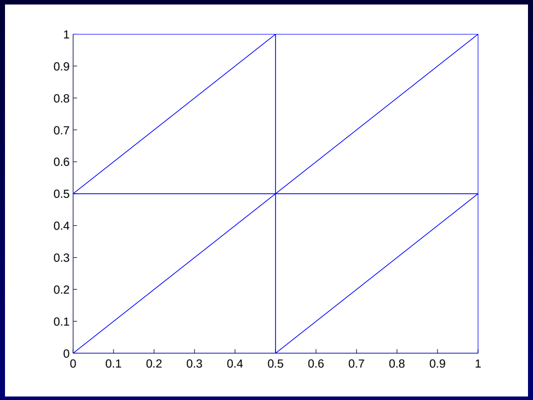
Finite element spaces

- K is a closed subset of $\mathbb{R}^2$ with a nonempty interior and a Lipshitz continuous boundary
- P_K is a finite dimensional space of vector valued or matrix valued functions defined over the set K
- Θ_K is a finite set of linearly independent linear functionals, $\theta_i, i = 1, \ldots, N$ referred to as degrees of freedom of the finite element, defined over the set P_K.

It is assumed that the set Θ_K is P_K-unisolvent in the sense that

$$\theta_i(p) = 0, i = 1, \ldots, N \Longrightarrow p \equiv 0$$

Examples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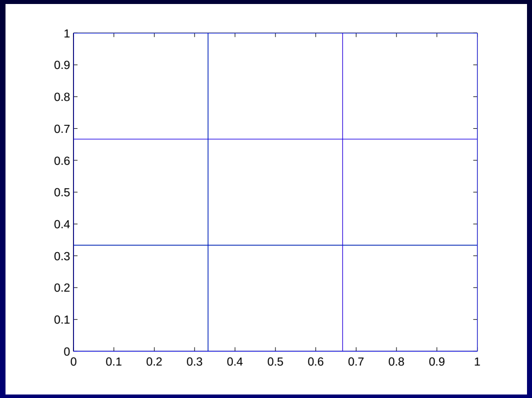
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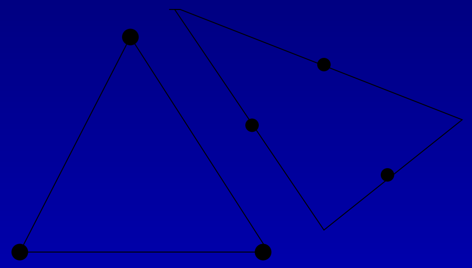

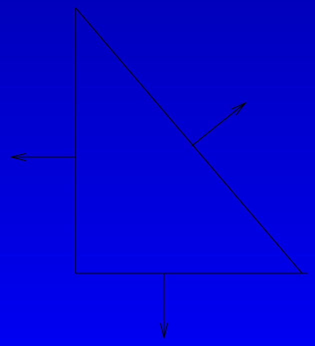

Brezzi's stability conditions

$\Sigma_h \subset \Sigma$ and $V_h \subset V$

Sufficient conditions for optimal error bounds

First Brezzi condition $\exists \alpha > 0$ independent of h such that

$$a(\tau, \tau) \geq \alpha \|\tau\|_\Sigma^2$$

for all τ in K_h where

$$K_h = \{\tau \in \Sigma_h : b(\tau, v) = 0, \forall v \in V_h\}$$

Second Brezzi condition $\exists \beta > 0$ independent of h such that

$$\sup_{\tau \in \Sigma_h} \frac{b(\tau, v)}{\|\tau\|_\Sigma} \geq \beta \|v\|_V \quad \forall v \in V_h$$

$$\|\sigma - \sigma_h\|_\Sigma + \|u - u_h\|_V \leq \gamma \{\inf_{\tau \in \Sigma_h} \|\sigma - \tau\|_\Sigma + \inf_{v_h \in V_h} \|u - v_h\|_V\}$$

with γ independent of h.

Surprisingly stable mixed finite elements for elasticity have been difficult to construct.

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Modified variational problems Arnold, Brezzi and Douglas (1984), Stenberg (1986), Arnold and Falk(1988), Morley (1989), Becache, Joly and Tsogka (2002), Arnold, Falk and Winther (2005).... However, none of these elements seemed to be the perfect choice.

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Using polynomial shape functions D. Arnold and R. Winther, (2002), D. Arnold and R. Winther, (2003), S. Adams and B. Cockburm, (2004).

My own contributions:

D. Arnold, G. Awanou and R. Winther, A Family of Mixed Elements for Elasticity in three Dimensions: (In preparation, 2005).

D. Arnold and G. Awanou, Rectangular Mixed Finite Elements for Elasticity, Submitted 2005,

G. Awanou Nonconforming Rectangular Mixed Finite Elements for Elasticity, (In preparation 2005).

G. Awanou Three dimensional Rectangular Mixed Finite Elements for Elasticity, (In preparation 2005).

Commutative Diagram

Sufficient conditions for stability

- div $\Sigma_h \subset V_h$.
- There exists a linear operator $\Pi_h : H^1(\Omega, \mathrm{Sym}) \to \Sigma_h$, bounded in $\mathcal{L}(H^1, L^2)$ uniformly with respect to h, and such that with $P_h : L^2(\Omega, \mathbb{R}^2) \to V_h$ denoting the L^2-projection

$$\begin{tikzcd}
H(\mathrm{div}, \Omega, \mathrm{Sym}) \arrow[r, "\mathrm{div}"] \arrow[d, "\pi_h"] & L^2(\Omega, \mathbb{R}^2) \arrow[d, "P_h"] \\
\Sigma_h \arrow[r, "\mathrm{div}"] & V_h
\end{tikzcd}$$

Elasticity Differential Complex

$$\begin{tikzcd}
\mathcal{P}_1(\Omega) \arrow[r, "\subset"] & H^2(\Omega) \arrow[r, "J"] \arrow[d, "I_h"] & H(\mathrm{div}, \Omega, \mathrm{Sym}) \arrow[r, "\mathrm{div}"] \arrow[d, "\pi_h"] & L^2(\Omega, \mathbb{R}^2) \arrow[r] \arrow[d, "P_h"] & 0 \\
\mathcal{P}_1(\Omega) \arrow[r, "\subset"] & Q_h \arrow[r, "J"] & \Sigma_h \arrow[r, "\mathrm{div}"] & V_h \arrow[r] & 0
\end{tikzcd}$$

$$Jq := \begin{pmatrix} \dfrac{\partial^2 q}{\partial y^2} & -\dfrac{\partial^2 q}{\partial x \partial y} \\ -\dfrac{\partial^2 q}{\partial x \partial y} & \dfrac{\partial^2 q}{\partial x^2} \end{pmatrix}$$

Rectangular elements

$$\mathcal{P}_1(R) \xrightarrow{\subset} \mathcal{P}_{5,5}(R) \xrightarrow{J} \begin{pmatrix} \mathcal{P}_{5,3} & \mathcal{P}_{4,4} \\ \mathcal{P}_{4,4} & \mathcal{P}_{3,5} \end{pmatrix}_{\mathbb{S}} \xrightarrow{\text{div}} \begin{pmatrix} \mathcal{P}_{4,3} \\ \mathcal{P}_{3,4} \end{pmatrix} \rightarrow 0$$

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$$\Sigma_R = \left\{ \tau \; \in \begin{pmatrix} \mathcal{P}_{5,3} & \mathcal{P}_{4,4} \\ \mathcal{P}_{4,4} & \mathcal{P}_{3,5} \end{pmatrix}_{\mathbb{S}}, \ \text{div}\ \tau \in V_R \right\} \qquad \text{and}$$

$$V_R = \begin{pmatrix} \mathcal{P}_{2,1} \\ \mathcal{P}_{1,2} \end{pmatrix}$$

Degrees of Freedom of Σ_R

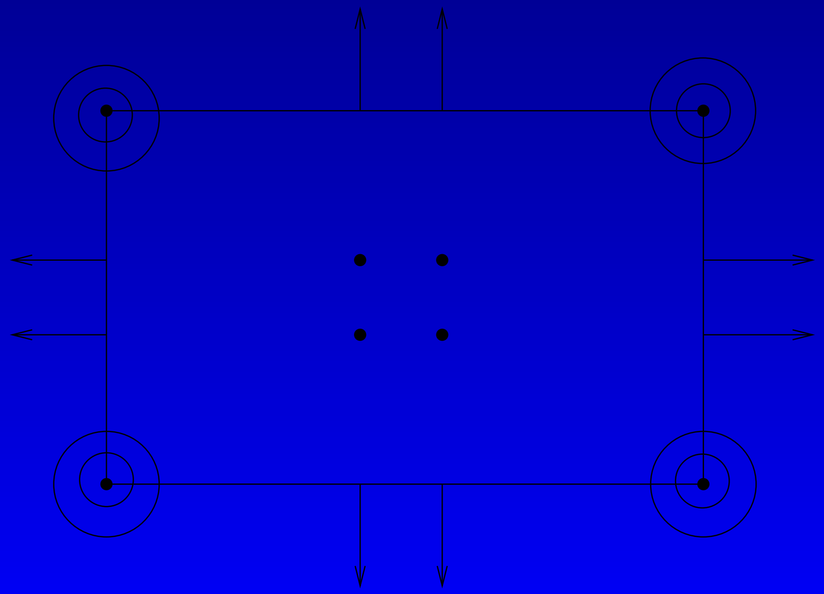
1. the values of each component of $\tau(x)$ at the vertices of R (12 degrees of freedom)
2. the first two moments of $(\tau n) \cdot n$ on each edge (8 degrees of freedom)
3. the first three moments of $(\tau n) \cdot t$ on each edge (12 degrees of freedom)
4. the values of $\int_R \tau : \phi$ for all ϕ in $\epsilon(V_R)$ (9 degrees of freedom)
5. the values of $\int_R \tau : \phi$ for all ϕ in $M_1(R)$ (4 degrees of freedom)

$$M_1(R) := \left\{ \tau \ \in \begin{pmatrix} \mathcal{P}_{5,3} & \mathcal{P}_{4,4} \\ \mathcal{P}_{4,4} & \mathcal{P}_{3,5} \end{pmatrix}_{\mathbb{S}}, \ \operatorname{div} \tau = 0 \text{ and } \tau n = 0 \text{ on } \partial R \right\}$$

H^2 element

We take $Q_R = \mathcal{P}_{5,5}(R)$ with the following 36 degrees of freedom

1. derivatives up to order 2 at each vertex ($6 \times 4 = 24$ degrees of freedom)
2. moments of degree 0 and 1 of $\partial q/\partial n$ on each edge ($2 \times 4 = 8$ degrees of freedom)
3. $\int_R J(q) : \phi \, dx, \quad$ for all $\phi \in M_1$ (4 degrees of freedom)



Discrete Elasticity Sequence

$$\begin{tikzcd}
\mathcal{P}_1(\Omega) \arrow[r, "\subset"] & C^\infty(\Omega) \arrow[r, "J"] \arrow[d, "I_h"] & C^\infty(\Omega, \mathbb{S}) \arrow[r, "\mathrm{div}"] \arrow[d, "\pi_h"] & C^\infty(\Omega, \mathbb{R}^2) \arrow[r] \arrow[d, "P_h"] & 0 \\
\mathcal{P}_1(\Omega) \arrow[r, "\subset"] & Q_h \arrow[r, "J"] & \Sigma_h \arrow[r, "\mathrm{div}"] & V_h \arrow[r] & 0
\end{tikzcd}$$

Error estimates

$$||\sigma - \sigma_h||_0 \leq ch^2 ||\sigma||_3$$

$$||u - u_h||_0 \leq ch^2 ||u||_2$$

Extensions

Stable Higher order elements have been constructed.

Simplified version of the lowest order elements

Nonconforming versions of these elements

Extended to three dimensions on tetrahedral meshes

Extension to three dimensional rectangular meshes

Higher order elements $k \geq 1$

$V_R = \begin{pmatrix} \mathcal{P}_{k+1,k} \\ \mathcal{P}_{k,k+1} \end{pmatrix}$ dim $\Sigma_R = 3k^2 + 16k + 26$ with

$$\Sigma_R = \left\{ \tau \ \in \begin{pmatrix} \mathcal{P}_{k+4,k+2} & \mathcal{P}_{k+3,k+3} \\ \mathcal{P}_{k+3,k+3} & \mathcal{P}_{k+2,k+4} \end{pmatrix}_{\mathbb{S}}, \ \text{div}\ \tau \in V_R \right\}$$

O(h^{k+1}) for stress and displacement

Simplified element of low order

RM(R): space of infinitesimal rigid motions, $(a - cx_2, b - cx_1)$

$V_R = RM(R)$ dim $\Sigma_R = 36$ with

$$\Sigma_R = \left\{ \tau \ \in \begin{pmatrix} \mathcal{P}_{5,3} & \mathcal{P}_{4,4} \\ \mathcal{P}_{4,4} & \mathcal{P}_{3,5} \end{pmatrix}_{\mathbb{S}} \ \text{and div}\ \tau \in RM(R) \right\}$$

O(h) for stress and displacement

Nonconforming elements

$$V_R = \begin{pmatrix} \mathcal{P}_{2,1} \\ \mathcal{P}_{1,2} \end{pmatrix} \text{ dim } \Sigma_R = 30 \text{ with}$$

$$\Sigma_R = \left\{ \tau \ \in \begin{pmatrix} \mathcal{P}_{4,2} & \mathcal{P}_{3,3} \\ \mathcal{P}_{3,3} & \mathcal{P}_{2,4} \end{pmatrix}_{\mathbb{S}}, \tau n \cdot n \in \mathcal{P}_1(e, \mathbb{R}) \right.$$
$$\left. \text{for each edge } e \text{ of } R, \text{ and div } \tau \in V_R \right\}$$

For $V_R = RM(R)$ dim $\Sigma_R = 21$

O(h) for stress and displacement

Tetrahedral elements

$$\begin{aligned} \mathcal{T} \quad &\to \quad H^1(\Omega,\mathbb{R})^3 \xrightarrow{\epsilon} H(\mathrm{curl\ curl}^*,\Omega,\mathrm{Sym}) \\ &\xrightarrow{\mathrm{curl\ curl}^*} H(\mathrm{div},\Omega,\mathrm{Sym}) \xrightarrow{\mathrm{div}} L^2(\Omega,\mathbb{R}^3) \to 0 \end{aligned}$$

$$\begin{aligned} \mathcal{T} \quad &\xrightarrow{\subseteq} \quad \mathcal{P}_{k+4}(\Omega,\mathbb{R}^3) \xrightarrow{\epsilon} \mathcal{P}_{k+3}(\Omega,\mathrm{Sym}) \\ &\xrightarrow{\mathrm{curl\ curl}^*} \mathcal{P}_{k+1}(\Omega,\mathrm{Sym}) \xrightarrow{\mathrm{div}} \mathcal{P}_k(\Omega,\mathbb{R}^3) \to 0. \end{aligned}$$

$$\mathcal{T} \to R_h \xrightarrow{\epsilon} Q_h \xrightarrow{\mathrm{curl\ curl}^*} \Sigma_h \xrightarrow{\mathrm{div}} V_h \to 0$$

Features of the elements

V_R space of discontinuous piecewise polynomials

Σ_R space of matrix fields with degrees of freedom

- vertex degrees of freedom
- degrees of freedom for τn
- $\int_R \tau : \epsilon(v), v \in V_R$
- $\int_R \tau : \phi,\ \phi$ in

$$\{\tau \in \Sigma_R,\ \text{div}\ \tau = 0, \tau n = 0 \text{ on } \partial R\}$$

Related future problems

- nonconforming tetrahedral elements for three dimensional elasticity without vertex degrees of freedom
- three dimensional rectangular elements
- quadrilateral elements
- unified analysis of the Arnold-Winther elements perhaps based on differential forms
- efficient computations for the discrete problem by iterative methods or parallel computations
- compatible discretizations (hot topic)

Summary

Elasticity differential complexes provide a guiding tool to design stable mixed elements.

Stable mixed elements for plane elasticity on rectangular meshes have been exhibited