MULTISCALE ASYMPTOTICS OF PARTIAL HEDGING

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ABSTRACT. We consider the problem of partial hedging of an European derivative under the assumption that the volatility is stochastic, driven by two diffusion processes, one fast mean reverting and the other varying slowly. For options with long maturities typically beyond 90 days, the singular perturbation analysis in [Partial Hedging in a Stochastic Volatility Environment, M. Jonsson and K.R. Sircar, Mathematical Finance, 12, pp. 375-409, 2002] ignores the slow factor. In this paper, we investigate the full two factors model and show how an additional term can be included in the approximate value functions and strategies.

1. Introduction

We consider the problem of shortfall risk minimization in stochastic volatility models under the assumption that the volatility is driven by two diffusions, one slowly varying and one fast mean reverting. Following the methodology of [3], the shortfall risk minimization problem is transformed into a state dependent utility maximization problem. The optimal strategies depend on the solution of a high dimensional HJB equation satisfied by the value function of the utility problem. The PDE satisfied by the Legendre transform of the value function is derived along with the one satisfied by the smallest optimizer in the Legendre transform. That optimizer can be thought as the inverse of marginal utility and is shown to be the price of a modified claim. We perform the asymptotical analysis on the inverse of the marginal utility and derive approximate strategies and value functions.

2. Multiscale stochastic volatility models

Let $(\Omega, \mathcal{F}, P)$ denote a probability space which describes a financial market with time horizon T and a risky asset whose price $X_t \equiv X(t)$ satisfies

$$dX_t = \mu X_t \, dt + \sigma_t X_t \, dW_t^{(0)},$$

where $W_t^{(0)}$ is a standard one-dimensional Brownian motion, μ and σ_t are the mean and volatility of the stock respectively. We consider in this paper a two-factor stochastic volatility model

$$\sigma_t = f(Y_t, Z_t)$$

for a smooth, bounded and positive function f which is also bounded away from zero.

The process Y_t is a fast mean reverting diffusion process. To fix ideas, we shall assume that it is a Gaussian Ornstein-Uhlenbeck process with rate of mean reversion α and invariant distribution a Gaussian with mean m and variance $\nu = \beta^2/2\alpha$;

$$dY_t = \alpha(m - Y_t) \, dt + \beta \, dW_t^{(1)},$$

where $W_t^{(1)}$ is a standard one-dimensional Brownian motion with

$$d\langle W_t^{(0)}, W_t^{(1)}\rangle = \rho_1\, dt$$

and ρ_1 is constant, $|\rho_1| < 1$. Put $\alpha = \frac{1}{\epsilon}$ so that $\beta = \sqrt{2}\nu/\sqrt{\epsilon}$, $\epsilon > 0$ being the time scale of the process. Fast mean reversion occurs when ϵ is small and the variance ν is moderate.

The process Z_t is a slowly varying diffusion process

$$dZ_t = \delta c(Z_t)\, dt + \sqrt{\delta} h(Z_t)\, dW_t^{(2)},$$

where $\delta > 0$ is a small parameter, c and g are smooth and at most linearly growing at infinity. We assume that

$$\begin{aligned} d\langle W_t^{(0)}, W_t^{(2)}\rangle &= \rho_2\, dt \\ d\langle W_t^{(1)}, W_t^{(2)}\rangle &= \rho_{12}\, dt, \end{aligned}$$

with constants ρ_2 and ρ_{12}. More explicitly, the correlation between the three Brownian motions can be described as

$$\begin{aligned} dW_t^{(0)} &= dB_t^1 \\ dW_t^{(1)} &= \rho_1\, dB_t^1 + \sqrt{1-\rho_1^2}\, dB_t^2 \\ dW_t^{(2)} &= \rho_1\, dB_t^1 + \overline{\rho_{12}}\, dB_t^2 + \sqrt{1-\rho_2^2-\overline{\rho_{12}}^2}\, dB_t^3 \end{aligned}$$

where $\overline{\rho_{12}}$ is a constant which satisfy $\rho_2^2 + \overline{\rho_{12}}^2 < 1$ and (B_t^1, B_t^2, B_t^3) is a standard three dimensional Brownian motion. Notice that we have $\rho_{12} = \rho_1\rho_2 + \overline{\rho_{12}}\sqrt{1-\rho_1^2}$. In summary we have

$$\begin{aligned} dX_t &= \mu X_t\, dt + \sigma_t X_t\, dW_t^{(0)} \\ dY_t &= \alpha(m - Y_t)\, dt + \beta\, dW_t^{(1)} \\ dZ_t &= \delta c(Z_t)\, dt + \sqrt{\delta} h(Z_t)\, dW_t^{(2)} \end{aligned} \tag{1}$$

3. Shortfall risk minimization

Let $\mathcal{F}_t, 0 \leq t \leq T$ be the σ-field generated by $\{X_u, Y_u, Z_u, 0 \leq u \leq t\}$. For simplicity, we shall assume that $\mathcal{F}_0$ is trivial and $\mathcal{F}_T = \mathcal{F}$. Let also g be a continuous function from $\mathbb{R}$ into $\mathbb{R}^+$. We consider an investor who has an initial capital $v \geq 0$ and is willing to invest π_t dollars in the stock. His goal is to hedge a contingent claim $H = g(X_T)$, and we shall assume that he is using self-financing strategies so that the value V_t of his portfolio is given by

$$V_t = v + \int_0^t \frac{\pi_u}{X_u}\, dX_u, \qquad 0 \leq t \leq T.$$

We require $\pi_t, 0 \leq t \leq T$ predictable with π_t/X_t integrable. If the initial capital is not high enough, there is a possibility of shortfall, $V_T < H$. For a given initial value x of the risky asset, we are interested in strategies which minimize the shortfall risk

$$\mathbb{E}\{l(H - V_T)^+\}$$

under the constraint $V_t \geq 0$ for all t, where l is a positive convex function with $l(0) = 0$ and $x^+ = \max(x, 0)$. To fix ideas we shall assume that $l(u) = \frac{1}{2}u^2$. Let us define the state dependent utility function

$$U(x, v) = \frac{1}{2}[h(x)^2 - ((h(x) - v)^+)^2].$$

It is easy to see that the problem is equivalent to maximize $\mathbb{E}\{U(X_T, V_T)\}$ under the constraint $V_t \geq 0$ for all t.

HJB equation. Recall (1), the dynamics of (X_t, Y_t, Z_t) and notice that the portfolio value process satisfies

$$dV_t = \pi_t \mu \, dt + \pi_t f(Y_t, Z_t) \, dW_t^{(0)}.$$

We next define

$$H(t, x, y, z, v) = \sup_\pi \mathbb{E}_{t,x,y,z,v}\{U(X_T, V_T)\}$$

where $\mathbb{E}_{t,x,y,z,v}$ is the expectation conditional to $X_0 = x, Y_0 = y, Z_0 = z, V_0 = v$. We also introduce the infinitesimal generator

$$\begin{aligned}\mathcal{L}_{x,y,z}H = {} & \mu x H_x + \alpha(m - y)H_y + \delta c(z)H_z + \frac{1}{2}x^2 f(y, z)^2 H_{xx} \\ & + \beta\rho_1 x f(y, z)H_{xy} + \sqrt{\delta}h(z)\rho_2 x f(y, z)H_{xz} + \frac{1}{2}\beta^2 H_{yy} \\ & + \sqrt{\delta}h(z)\beta\rho_{12}H_{zy} + \frac{1}{2}\delta h(z)^2 H_{zz}.\end{aligned} \tag{2}$$

By the Bellman principle, H satisfies the following HJB equation

$$\begin{aligned}H_t + \mathcal{L}_{x,y,z}H + \max_\pi\{ & \pi\mu H_v + \pi x f(y, z)^2 H_{xv} + \beta\rho_1 \pi f(y, z)H_{vy} \\ & + \sqrt{\delta}h(z)\rho_2 \pi f(y, z)H_{zv} + \frac{1}{2}\pi^2 f(y, z)^2 H_{vv}\} = 0.\end{aligned}$$

It is not difficult to find that the maximum occurs at

$$\pi^* = -\frac{\mu H_v + x f(y, z)^2 H_{xv} + \beta\rho_1 f(y, z)H_{vy} + \sqrt{\delta}h(z)\rho_2 f(y, z)H_{zv}}{f(y, z)^2 H_{vv}} \tag{3}$$

so

$$H_t + \mathcal{L}_{x,y,z}H - \frac{[\mu H_v + f(y, z)^2 x H_{xv} + \rho_1\beta f(y, z)H_{vy} + \sqrt{\delta}h(z)\rho_2 f(y, z)H_{zv}]^2}{2f(y, z)^2 H_{vv}} = 0$$

in the domain $x > 0, -\infty < y < \infty, -\infty < z < \infty, v > 0$ and $t < T$ and terminal condition

$$H(T, x, y, z, v) = U(x, v).$$

Legendre transform. We now use the Legendre transform to transform this nonlinear PDE to a PDE with several linear terms. Let us assume that H is convex in v and define the convex dual $\hat{H}$ of H with respect to v by

$$\hat{H}(t, x, y, z, r) = \sup_{v>0}\{H(t, x, y, z, v) - rv\}. \tag{4}$$

We also define

$$g(t, x, y, z, r) = \inf\{v > 0 \mid H(t, x, y, z, v) \geq rv + \hat{H}(t, x, y, z, r)\} \tag{5}$$

which is in some sense the smallest optimizer in (4). If H is sufficiently smooth, $g = H_v^{-1}$, so that g is the inverse of the marginal utility. We have the following equations which relate H to $\hat{H}$.

$$\begin{aligned} \hat{H}(t,x,y,z,r) &= -rg(t,x,y,z,r) + H(t,x,y,z,g(t,x,y,z,r)) \\ H(t,x,y,z,v) &= v\hat{g}(t,x,y,z,v) + \hat{H}(t,x,y,z,\hat{g}(t,x,y,z,v)) \end{aligned}$$

$$\begin{aligned} &H_v(t,x,y,z,g(t,x,y,z,r)) = r && H_v(t,x,y,z,v) = \hat{g}(t,x,y,z,v) \\ &\hat{H}_r(t,x,y,z,\hat{g}(t,x,y,z,v)) = -v && \hat{H}_r(t,x,y,z,r) = -g(t,x,y,z,r) \end{aligned}$$

$$H_v = r \quad H_t = \hat{H}_t \quad H_x = \hat{H}_x \quad H_y = \hat{H}_y \quad H_z = \hat{H}_z$$

$$H_{vv} = -\frac{1}{\hat{H}_{rr}} \quad H_{xv} = -\frac{\hat{H}_{xr}}{\hat{H}_{rr}} \qquad H_{yv} = -\frac{\hat{H}_{yr}}{\hat{H}_{rr}} \qquad H_{zv} = -\frac{\hat{H}_{zr}}{\hat{H}_{rr}}$$

$$\begin{aligned} &H_{xx} = \hat{H}_{xx} - \frac{\hat{H}_{xr}^2}{\hat{H}_{rr}} && H_{yy} = \hat{H}_{yy} - \frac{\hat{H}_{yr}^2}{\hat{H}_{rr}} && H_{zz} = \hat{H}_{zz} - \frac{\hat{H}_{zr}^2}{\hat{H}_{rr}} \\ &H_{xy} = \hat{H}_{xy} - \frac{\hat{H}_{xr}\hat{H}_{yr}}{\hat{H}_{rr}} && H_{xz} = \hat{H}_{xz} - \frac{\hat{H}_{xr}\hat{H}_{zr}}{\hat{H}_{rr}} && H_{yz} = \hat{H}_{yz} - \frac{\hat{H}_{yr}\hat{H}_{zr}}{\hat{H}_{rr}} \end{aligned}$$

From the PDE of H and the above relationships between H and $\hat{H}$, we get the following equations for $\hat{H}$,

$$\begin{aligned} &\hat{H}_t + \mathcal{L}_{x,y,z}\hat{H} - \mu r x \hat{H}_{xr} - \frac{\mu r \rho_1 \beta}{f(y,z)}\hat{H}_{yr} - \frac{\mu r \sqrt{\delta} h(z)\rho_2}{f(y,z)}\hat{H}_{zr} \\ &\qquad + \frac{1}{2}\frac{\mu^2 r^2}{f(y,z)^2}\hat{H}_{rr} - \frac{1}{2}\beta^2(1-\rho_1^2)\frac{\hat{H}_{yr}^2}{\hat{H}_{rr}} - \frac{1}{2}\delta h(z)^2(1-\rho_2^2)\frac{\hat{H}_{zr}^2}{\hat{H}_{rr}} \\ &\qquad - \sqrt{1-\rho_1^2}\,\overline{\rho}_{12}\beta\sqrt{\delta}h(z)\frac{\hat{H}_{yr}\hat{H}_{zr}}{\hat{H}_{rr}} = 0, \end{aligned} \tag{6}$$

with terminal condition

$$\begin{aligned} \hat{H}(T,x,y,z,r) &= \sup_{v>0}\{H(T,x,y,z,v) - rv\} \\ &= \sup_{v>0}\{U(x,v) - rv\} \\ &:= \hat{U}(x,v). \end{aligned}$$

The PDE for g is obtained from the one of $\hat{H}$, by differentiation and using

$$g(t,x,y,z,r) = -\hat{H}_r(t,x,y,z,r).$$

We have

$$
\begin{aligned}
& g_t+\mathcal{L}_{x,y,z}g - \mu x g_x - \frac{\rho_1\beta\mu}{f(y,z)} g_y - \frac{\mu\sqrt{\delta}h(z)\rho_2}{f(y,z)} g_z + \frac{\mu^2}{f(y,z)^2} r^2 g_r \\
& -\mu x r g_{xr} - \frac{\rho_1\beta\mu}{f(y,z)} r g_{yr} - \beta^2(1-\rho_1^2)\frac{g_y}{g_r} g_{yr} \\
& -\sqrt{1-\rho_1^2}\,\overline{\rho}_{12}\beta\sqrt{\delta}h(z)\frac{g_z}{g_r} g_{yr} - \frac{\mu r\sqrt{\delta}h(z)\rho_2}{f(y,z)} g_{zr} - \delta h(z)^2(1-\rho_2^2)\frac{g_z}{g_r} g_{zr} \\
& -\sqrt{1-\rho_1^2}\,\overline{\rho}_{12}\beta\sqrt{\delta}h(z)\frac{g_y}{g_r} g_{zr} + \frac{\mu^2}{2f(y,z)^2} r^2 g_{rr} + \frac{1}{2}(1-\rho_1^2)\beta^2\frac{g_y^2}{g_r^2} g_{rr} \\
& +\frac{1}{2}\delta h(z)^2(1-\rho_2^2)\frac{g_z^2}{g_r^2} g_{rr} + \sqrt{1-\rho_1^2}\,\overline{\rho}_{12}\beta\sqrt{\delta}h(z)\frac{g_y g_z}{g_r^2} g_{rr} = 0,
\end{aligned}
\tag{7}
$$

with terminal condition

$$
\begin{aligned}
g(T,x,y,z,r) = &\ \inf_{v>0}\{U(x,v) \geq rv + \hat{U}(x,r)\} \\
:= &\ G(x,r).
\end{aligned}
$$

Hedging strategies.

Using the expression of the optimal strategy (3), relations between the derivatives of $\hat{H}$ and H, one can write π^* in terms of g and then get the optimal strategies. Explicitly

$$
\pi^* = -\frac{\mu}{f(y,z)^2} r g_r + x g_x + \frac{\beta\rho_1}{f(y,z)} g_y + \frac{\sqrt{\delta}h(z)\rho_2}{f(y,z)} g_z. \tag{8}
$$

Pricing a modified claim. It can be shown that (6) is the HJB equation for the control problem

$$
\hat{H}(t,x,y,z,r) = \inf_{\gamma,\eta}\mathbb{E}_{t,x,y,z,r}\{\hat{U}(X_T,Y_T,Z_T,R_T^{\gamma,\eta})\},
$$

where (X_t, Y_t, Z_t) satisfy (1) and $R_t^{\gamma,\eta}$ is defined by

$$
dR_t^{\gamma,\eta} = -\frac{\mu}{f(Y_t,Z_t)} R_t^{\gamma,\eta}\, dB_t^1 + \gamma_t\, dB_t^2 + \eta_t\, dB_3^t.
$$

Moreover it is possible to choose γ and η so that the optimal strategy is the perfect hedge of the claim $G(X_T, R_T^{\gamma,\eta})$. To see this, let us assume that at time t, the price of the claim is given by $\tilde{g}(t, X_t, Y_t, Z_t, R_t^{\gamma,\eta})$ for a smooth function $\tilde{g}$ to be determined with $\tilde{g}(T,x,y,z,r) = G(x,r)$. We then consider the portfolio which consists of the claim and $-\Delta_t$ units of the the stock at time t

$$
\Pi_t = \tilde{g}(t, X_t, Y_t, Z_t, R_t^{\gamma,\eta}) - \Delta_t X_t.
$$

Using Itô's Lemma, we get an expression for Δ_t, γ_t and η_t for the portfolio to be instanteneously riskless in terms of $\tilde{g}$ and a PDE for $\tilde{g}$ which turns out to be the same as the one of g. Since they have the same terminal condition, they are equal.

Explicitly, we have

$$\Delta_t = g_x + \frac{\beta\rho_1}{X_t f(Y_t,Z_t)} g_y + \frac{\sqrt{\delta} h(Z_t)\rho_2}{X_t f(Y_t,Z_t)} g_z$$
$$\gamma_t = -\beta\sqrt{1-\rho_1^2}\frac{g_y}{g_r} + \sqrt{\delta} h(Z_t)\overline{\rho_{12}}\frac{g_z}{g_r}$$
$$\eta_t = -\sqrt{\delta} h(Z_t)\sqrt{1-\rho_2^2-\overline{\rho_{12}}^2}\frac{g_z}{g_r}.$$

4. Multiscale asymptotics

Recall that $\alpha = \frac{1}{\epsilon}$ and $\beta = \sqrt{2}\nu/\sqrt{\epsilon}$. We first define the following linear and nonlinear operators

$$\begin{aligned}
\mathcal{L}_0 &= \nu^2 \frac{\partial^2}{\partial y^2} + (m-y)\frac{\partial}{\partial y} \\
\mathcal{L}_1 &= \sqrt{2}\rho_1\nu\left(f(y,z)x\frac{\partial^2}{\partial x \partial y} - \frac{\mu}{f(y,z)} r \frac{\partial^2}{\partial y \partial r} - \frac{\mu}{f(y,z)}\frac{\partial}{\partial y}\right) \\
\mathcal{L}_2 &= \frac{\partial}{\partial t} + \frac{1}{2} f(y,z)^2 x^2 \frac{\partial^2}{\partial x^2} + \frac{\mu^2}{f(y,z)^2}\left(\frac{1}{2} r^2 \frac{\partial^2}{\partial r^2} + r\frac{\partial}{\partial r}\right) \\
&\qquad - \mu x r \frac{\partial^2}{\partial x \partial r}
\end{aligned} \tag{9}$$

$$\begin{aligned}
\mathcal{M}_1 &= -\frac{\mu\rho_2 h(z)}{f(y,z)}\frac{\partial}{\partial z} - \frac{\mu\rho_2 h(z)}{f(y,z)} r \frac{\partial^2}{\partial z \partial r} + h(z)\rho_2 x f(y,z)\frac{\partial^2}{\partial x \partial z} \\
\mathcal{M}_2 &= c(z)\frac{\partial}{\partial z} + \frac{h(z)^2}{2}\frac{\partial^2}{\partial z^2} \\
\mathcal{M}_3 &= \sqrt{2}\nu\rho_{12} h(z)\frac{\partial^2}{\partial z \partial y}
\end{aligned} \tag{10}$$

$$\begin{aligned}
\mathcal{NL}_{yr}(g) &= -\left(\frac{2 g_{yr} g_y}{g_r} - \frac{g_y^2}{g_r^2} g_{rr}\right) = -\frac{\partial}{\partial r}\left(\frac{g_y^2}{g_r}\right) \\
\mathcal{NL}_{zr}(g) &= -\left(\frac{2 g_{zr} g_z}{g_r} - \frac{g_z^2}{g_r^2} g_{rr}\right) = -\frac{\partial}{\partial r}\left(\frac{g_z^2}{g_r}\right) \\
\mathcal{NL}_{yzr}(g) &= -\left(\frac{g_{yr} g_z + g_{zr} g_y}{g_r} - \frac{g_y g_z}{g_r^2} g_{rr}\right) = -\frac{\partial}{\partial r}\left(\frac{g_y g_z}{g_r}\right)
\end{aligned} \tag{11}$$

Using (2) and (7), g is seen to satisfy the PDE

$$\begin{aligned}
&\left(\frac{1}{\epsilon}\mathcal{L}_0 + \frac{1}{\sqrt{\epsilon}}\mathcal{L}_1 + \mathcal{L}_2\right) g + \left(\sqrt{\delta}\mathcal{M}_1 + \delta\mathcal{M}_2 + \sqrt{\frac{\delta}{\epsilon}}\mathcal{M}_3\right) g \\
&\quad + \frac{\nu^2}{\epsilon}(1-\rho_1^2)\mathcal{NL}_{yr}(g) + \frac{\delta h(z)^2}{2}(1-\rho_2^2)\mathcal{NL}_{zr}(g) \\
&\quad + \sqrt{2(1-\rho_1^2)}\overline{\rho}_{12}\nu\sqrt{\frac{\delta}{\epsilon}} h(z)\mathcal{NL}_{yzr}(g) = 0.
\end{aligned} \tag{12}$$

Let us put

$$\gamma_1 = \nu^2(1-\rho_1^2), \quad \gamma_2 = \frac{h(z)^2}{2}(1-\rho_2^2), \quad \text{and } \gamma_3 = \sqrt{2(1-\rho_1^2)}\overline{\rho}_{12}\nu h(z).$$

We next perform the asymptotics in the regime where ϵ and δ are small independent parameters. The term $g_{j,k}$ will be associated with the term of order $\epsilon^{\frac{j}{2}}\delta^{\frac{k}{2}}$.

Slow scale expansion.

We first make an expansion of $g \equiv g^{\epsilon,\delta}$ in powers of $\sqrt{\delta}$

$$g^{\epsilon,\delta} = g_0^\epsilon + \sqrt{\delta}g_1^\epsilon + \delta g_2^\epsilon + \dots \tag{13}$$

We will use several times the expansion

$$\begin{aligned}\mathcal{N}\mathcal{L}_{yr}(g^{\epsilon,\delta}) &= \mathcal{N}\mathcal{L}_{yr}(g_0^\epsilon) - \sqrt{\delta}\frac{\partial}{\partial_r}\left[\frac{\partial_y g_0^\epsilon}{\partial_r g_0^\epsilon}\left(2\partial_y g_1^\epsilon - \frac{\partial_y g_0^\epsilon}{\partial_r g_0^\epsilon}\partial_r g_1^\epsilon\right)\right] + \dots \\ &= \mathcal{N}\mathcal{L}_{yr}(g_0^\epsilon) - \sqrt{\delta}\frac{\partial}{\partial_r}\left[\partial_y g_0^\epsilon \; \Gamma(g_0^\epsilon, g_1^\epsilon)\right] + \dots,\end{aligned}$$

where $\Gamma(g_0^\epsilon, g_1^\epsilon) = \frac{1}{\partial_r g_0^\epsilon}\left(2\partial_y g_1^\epsilon - \frac{\partial_y g_0^\epsilon}{\partial_r g_0^\epsilon}\partial_r g_1^\epsilon\right)$. The order δ and $\delta^{\frac{3}{2}}$ terms are sums of terms which have a factor $\partial_y g_0^\epsilon$ and/or $\partial_y g_1^\epsilon$ and will not be needed in the terms we are interested in.

Substituing (13) into (12), and expanding in powers of δ we get from the first two terms,

$$\left(\frac{1}{\epsilon}\mathcal{L}_0 + \frac{1}{\sqrt{\epsilon}}\mathcal{L}_1 + \mathcal{L}_2\right)g_0^\epsilon + \frac{\gamma_1}{\epsilon}\mathcal{N}\mathcal{L}_{yr}(g_0^\epsilon) = 0, \tag{14}$$

$$\begin{aligned}&\left(\frac{1}{\epsilon}\mathcal{L}_0 + \frac{1}{\sqrt{\epsilon}}\mathcal{L}_1 + \mathcal{L}_2\right)g_1^\epsilon + \mathcal{M}_1 g_0^\epsilon + \frac{1}{\sqrt{\epsilon}}\mathcal{M}_3 g_0^\epsilon + \frac{1}{\sqrt{\epsilon}}\gamma_3\mathcal{N}\mathcal{L}_{yzr}(g_0^\epsilon) \\ &\qquad - \frac{\partial}{\partial_r}\left(\partial_y g_0^\epsilon \Gamma(g_0^\epsilon, g_1^\epsilon)\right) = 0.\end{aligned} \tag{15}$$

We append to these equations, the boundary conditions

$$\begin{aligned}&g_0^\epsilon(T,x,y,z,r) = g(T,x,y,z,r) = G(x,r) \\ &g_1^\epsilon(T,x,y,z,r) = 0\end{aligned}$$

Fast scale expansion.

We next expand g_0^ϵ and g_1^ϵ in powers of $\sqrt{\epsilon}$.

Expansion of g_0^ϵ.

$$g_0^\epsilon = g_0 + \sqrt{\epsilon}g_{1,0} + \epsilon g_{2,0} + \epsilon^{\frac{3}{2}}g_{3,0} + \dots \tag{16}$$

As above

$$\mathcal{N}\mathcal{L}_{yr}(g_0^\epsilon) = \mathcal{N}\mathcal{L}_{yr}(g_0) - \sqrt{\epsilon}\frac{\partial}{\partial_r}\left[\partial_y g_0 \; \Gamma(g_0, g_{1,0})\right] + \dots$$

Substituing into (14), we get

$$\frac{1}{\epsilon}\Big(\mathcal{L}_0 g_0 + \gamma_1 \mathcal{N}\mathcal{L}_{yr}(g_0)\Big) + \frac{1}{\sqrt{\epsilon}}\Big(\mathcal{L}_0 g_{1,0} + \mathcal{L}_1 g_0 - \gamma_1 \frac{\partial}{\partial_r}\big[\partial_y g_0 \ \Gamma(g_0, g_{1,0})\big]\Big) + \Big(\mathcal{L}_0 g_{2,0} + \mathcal{L}_1 g_{1,0} + \mathcal{L}_2 g_0\Big) + \sqrt{\epsilon}\Big(\mathcal{L}_0 g_{3,0} + \mathcal{L}_1 g_{2,0} + \mathcal{L}_2 g_{1,0}\Big) + \ldots = 0$$

From the lowest order term, we get

$$\mathcal{L}_0 g_0 + \gamma_1 \mathcal{N}\mathcal{L}_{yr}(g_0) = 0.$$

Notice that $\frac{1}{\epsilon}\mathcal{L}_0$ is the infinitesimal generator of the OU process Y_t. Arguing as in [3] p. 17, we conclude that g_0 does not depend on y. This implies that at the next order

$$\mathcal{L}_0 g_{1,0} + \mathcal{L}_1 g_0 = 0.$$

Since g_0 does not depend on y and $\mathcal{L}_1$ takes derivatives in y, $\mathcal{L}_1 g_0 = 0$. This gives $\mathcal{L}_0 g_{1,0} = 0$ from which we conclude that $g_{1,0}$ also does not depend on y. To see this, one can use the expression of κ' in [1] p. 93, where κ is the solution of $\mathcal{L}_0 \kappa = g$.

With $\mathcal{L}_1 g_{1,0} = 0$, we get from the zeroth order term

$$\mathcal{L}_0 g_{2,0} + \mathcal{L}_2 g_0 = 0.$$

We know from [1] p. 91, that a necessary condition for this Poisson equation to be solvable in $g_{2,0}$ is that

$$\langle \mathcal{L}_2 g_0 \rangle = 0,$$

where $\langle , \rangle$ denotes the average with respect to the invariant distribution of Y_t. The zeroth order approximation is then taken to be solution of

$$\begin{aligned} \langle \mathcal{L}_2 \rangle g_0 &= 0 \\ g_0(T, x, y, z, r) &= G(x, r), \end{aligned} \tag{17}$$

where

$$\langle \mathcal{L}_2 \rangle = \frac{\partial}{\partial t} + \frac{1}{2}\overline{\sigma}(z)^2 x^2 \frac{\partial^2}{\partial x^2} + \frac{\mu^2}{\sigma_*^2(z)}\Big(\frac{1}{2} r^2 \frac{\partial^2}{\partial r^2} + r\frac{\partial}{\partial r}\Big) - \mu x r \frac{\partial^2}{\partial x \partial r}$$

as g_0 does not depend on y and we have defined

$$\overline{\sigma}(z)^2 := \langle f^2(., z)\rangle \qquad \frac{1}{\sigma_*^2(z)} := \langle \frac{1}{f^2(., z)} \rangle. \tag{18}$$

Finally from the term of order $\sqrt{\epsilon}$, we have

$$\mathcal{L}_0 g_{3,0} + \mathcal{L}_1 g_{2,0} + \mathcal{L}_2 g_{1,0} = 0.$$

This is again a Poisson equation in $g_{3,0}$ and we have the centering condition

$$\langle \mathcal{L}_1 g_{2,0} + \mathcal{L}_2 g_{1,0} \rangle = 0.$$

Using a standard multiscale argument, e.g. [2] p. 8 for different operators, $g_{1,0}$ is seen as the solution of the problem

$$\begin{aligned} \langle \mathcal{L}_2 \rangle g_{1,0} &= \langle \mathcal{L}_1 \mathcal{L}_0^{-1}(\mathcal{L}_2 - \langle \mathcal{L}_2 \rangle)\rangle g_0 \\ g_{1,0}(T, x, y, z, r) &= 0. \end{aligned}$$

As in [3] p. 21-22, an explicit expression of $g_{1,0}$ can be given in terms of g_0 and certain market group parameters. In fact the problem is the same up to renaming the variables.

Expansion of g_1^ϵ.

$$g_1^\epsilon = g_{0,1} + \sqrt{\epsilon} g_{1,1} + \epsilon g_{2,1} + \epsilon^{\frac{3}{2}} g_{3,1} + \dots \tag{19}$$

Since g_0 does not depend on y, (15) becomes

$$\left(\frac{1}{\epsilon}\mathcal{L}_0 + \frac{1}{\sqrt{\epsilon}}\mathcal{L}_1 + \mathcal{L}_2\right) g_1^\epsilon + \mathcal{M}_1 g_0^\epsilon + \frac{1}{\sqrt{\epsilon}}\mathcal{M}_3 g_0^\epsilon + \frac{1}{\sqrt{\epsilon}}\gamma_3 \mathcal{N}\mathcal{L}_{yzr}(g_0^\epsilon) = 0. \tag{20}$$

Again since g_0 and $g_{1,0}$ do not depend on y, we have

$$\mathcal{N}\mathcal{L}_{yzr}(g_0^\epsilon) = O(\epsilon^2)$$

Therefore substituting (19) and (16) into (20) we obtain

$$\begin{aligned}\frac{1}{\epsilon}\mathcal{L}_0 g_{0,1} + \frac{1}{\sqrt{\epsilon}}\Big(\mathcal{L}_0 g_{1,1} + \mathcal{L}_1 g_{0,1} + \mathcal{M}_3 g_0\Big) \\ + \Big(\mathcal{L}_0 g_{2,1} + \mathcal{L}_1 g_{1,1} + \mathcal{L}_2 g_{0,1} + \mathcal{M}_1 g_0 + \mathcal{M}_3 g_{1,0}\Big) + \dots = 0\end{aligned}$$

The first order term gives

$$\mathcal{L}_0 g_{0,1} = 0$$

which implies as above that $g_{0,1}$ does not depend on y. Since $\mathcal{L}_1$ and $\mathcal{M}_3$ takes derivatives in y, we have at the next order

$$\mathcal{L}_0 g_{1,1} = 0$$

from which we conclude that $g_{1,1}$ also does not depend on y. The same arguments lead us to conclude that from the zeroth order term, the following equation holds

$$\mathcal{L}_0 g_{2,1} + \mathcal{L}_2 g_{0,1} + \mathcal{M}_1 g_0 = 0.$$

The solvability condition for this Poisson equation in $g_{2,1}$ leads us to conclude that $g_{0,1}$ solves

$$\langle \mathcal{L}_2 \rangle g_{0,1} = -\langle \mathcal{M}_1 \rangle g_0 \tag{21}$$

with zero terminal condition. This equation does not admit an exact solution and must be solved numerically. Notice that the term $\langle \mathcal{M}_1 \rangle$ involves $\langle f(y,z) \rangle$ and $\langle \frac{1}{f(y,z)} \rangle$ and so would require model specification.

Having obtained the first three terms in the asymptotics, we can use (8) to get the approximate hedging strategies.

5. Conclusion

The difference of this article with [3] is that we introduce an approximation term in the asymptotics to take into account a slow factor which invariably one would observe if one is interested in options with long maturities. To implement the method one should proceed as in [3].

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