1.2 Primes

from A Study Guide for Beginner’s by J.A.Beachy,
a supplement to Abstract Algebra by Beachy / Blair

Proposition 1.2.2 states that integers $a$ and $b$ are relatively prime if and only if there exist integers $m$ and $n$ with $ma + nb = 1$. This is one of the most useful tools in working with relatively prime integers. Remember that this only works in showing that $\gcd(a, b) = 1$. More generally, if you have a linear combination $ma + nb = d$, it only shows that $\gcd(a, b)$ is a divisor of $d$ (refer back to Theorem 1.1.6).

Since the fundamental theorem of arithmetic (on prime factorization) is proved in this section, you now have some more familiar techniques to use.

**SOLVED PROBLEMS: §1.2**

26. (a) Use the Euclidean algorithm to find $\gcd(1776, 1492)$.
   (b) Use the prime factorizations of 1776 and 1492 to find $\gcd(1776, 1492)$.

27. (a) Use the Euclidean algorithm to find $\gcd(1274, 1089)$.
   (b) Use the prime factorizations of 1274 and 1089 to find $\gcd(1274, 1089)$.

28. Give the diagram of all divisors of 250. Do the same for 484.

29. Find all integer solutions of the equation $xy + 2y - 3x = 25$.

30. Let $a, b, c$ be nonzero integers. Prove that if $b \mid a$ and $c \mid a$ and $\gcd(b, c) = d$, then $bc \mid ad$. Note: This extends Proposition 1.2.3 (c).

31. For positive integers $a, b, c$, prove that if $\gcd(a, b) = 1$ and $c \mid b$, then $\gcd(a, c) = 1$.

32. For positive integers $a, b$, prove that $\gcd(a, b) = 1$ if and only if $\gcd(a^2, b^2) = 1$.

33. Prove that $n - 1$ and $2n - 1$ are relatively prime, for all integers $n > 1$. Is the same true for $2n - 1$ and $3n - 1$?

34. Let $m$ and $n$ be positive integers. Prove that $\gcd(2^m - 1, 2^n - 1) = 1$ if and only if $\gcd(m, n) = 1$.

35. Prove that $\gcd(2n^2 + 4n - 3, 2n^2 + 6n - 4) = 1$, for all integers $n > 1$.

36. Prove that if $m$ and $n$ are odd integers, then $m^2 - n^2$ is divisible by 8. (Compare Problem 1.1.26.)

**MORE PROBLEMS: §1.2**

37.† Find the prime factorizations of 252 and 180 and use them to compute the greatest common divisor and least common multiple of 252 and 180.
   (Compare Problem 1.1.40.)
38.†Find the prime factorizations of 475 and 385 and use them to compute the greatest common divisor and least common multiple of 475 and 385.
   (Compare Problem 1.1.41.)

39.†Find the prime factorizations of 5917 and 4331 and use them to find \( \gcd(5917, 4331) \).
   (Compare Problem 1.1.43.)

40. Find the prime factorizations of 13651 and 3179 and use them to find \( \gcd(13651, 3179) \).
   (Compare Problem 1.1.44.)

41. Give a diagram of all divisors of 90, showing the divisibility relationships.

42.†Show that \( \gcd(11n + 5, 7n + 3) \) is 2 if \( n \) is odd and 1 if \( n \) is even.

43.†Find all positive integer solutions \( x, y \) of the equation \( xy + 5x - 8y = 79 \).

44. Explain why there are no positive integers \( x, y \) such that \( x^2 - y^2 = 34 \).

45. Let \( a, b, c \) be positive integers.
   (a) Prove that if \( \gcd(a, bc) = 1 \) and \( \gcd(b, c) = 1 \), then \( \gcd(ab, c) = 1 \).
   †(b) Prove or disprove the following generalization of part (a): if \( \gcd(b, c) = 1 \), then \( \gcd(a, bc) = \gcd(ab, c) \).

46. Let \( a, b, c \) be a Pythagorean triple (i.e. positive integers with \( a^2 + b^2 = c^2 \)).
   (a) Show that \( \gcd(a, b) = 1 \) if and only if \( \gcd(a, c) = 1 \).
   †(b) More generally, does \( \gcd(a, b) = \gcd(a, c) \)?