2.1 Functions

from A Study Guide for Beginner’s by J.A.Beachy,
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Besides reading Section 2.1, it might help to get out your calculus textbook and review composite functions, one-to-one and onto functions, and inverse functions. The functions $f : \mathbb{R} \to \mathbb{R}^+$ and $g : \mathbb{R}^+ \to \mathbb{R}$ defined by $f(x) = e^x$, for all $x \in \mathbb{R}$, and $g(y) = \ln y$, for all $y \in \mathbb{R}^+$, provide one of the most important examples of a pair of inverse functions.

Definition 2.1.1, the definition of function, is stated rather formally in terms of ordered pairs. (Think of this as a definition given in terms of the “graph” of the function.) This puts the definition on the firm foundation of set theory. But in actually using this definition, the text almost immediately goes back to what should be a more familiar definition: a function $f : S \to T$ is a “rule” that assigns to each element of $S$ a unique element of $T$.

One of the most fundamental ideas of abstract algebra is that algebraic structures should be thought of as essentially the same if the only difference between them is the way elements have been named. To make this precise we will say that structures are the same if we can set up an invertible function from one to the other that preserves the essential algebraic structure. That makes it especially important to understand the concept of an inverse function, as introduced in this section.

SOLVED PROBLEMS: §2.1

21. The “Vertical Line Test” from calculus says that a curve in the $xy$-plane is the graph of a function of $x$ if and only if no vertical line intersects the curve more than once. Explain why this agrees with Definition 2.1.1.

22. The “Horizontal Line Test” from calculus says that a function is one-to-one if and only if no horizontal line intersects its graph more than once. Explain why this agrees with Definition 2.1.4.

23. In calculus the graph of an inverse function $f^{-1}$ is obtained by reflecting the graph of $f$ about the line $y = x$. Explain why this agrees with Definition 2.1.6.

24. Show that the function $f : \mathbb{R}^2 \to \mathbb{R}^2$ defined by $f(x, y) = (x^3 + y, y)$, for all $(x, y) \in \mathbb{R}^2$, is a one-to-one correspondence.

25. Define $f : \mathbb{R} \to \mathbb{R}$ by $f(x) = x^3 + 3x - 5$, for all $x \in \mathbb{R}$. Is $f$ a one-to-one function? Is $f$ an onto function?

$Hint$: Use the derivative of $f$ to show that $f$ is a strictly increasing function.

26. Does the following formula define a function from $\mathbb{Q}$ to $\mathbb{Z}$? Set $f \left( \frac{m}{n} \right) = m$, where $m, n$ are integers and $n \neq 0$.

27. Define the formulas $f : \mathbb{Z}_{12} \to \mathbb{Z}_8$ by $f([x]_{12}) = [2x]_8$, for all $[x]_{12} \in \mathbb{Z}_{12}$, and $g : \mathbb{Z}_{12} \to \mathbb{Z}_8$ by $g([x]_{12}) = [3x]_8$, for all $[x]_{12} \in \mathbb{Z}_{12}$. Show that $f$ defines a function, but $g$ does not.
2.1 Let $a$ be a fixed element of $\mathbb{Z}_{17}^\times$. Define the function $\theta: \mathbb{Z}_{17}^\times \to \mathbb{Z}_{17}^\times$ by $\theta(x) = ax$, for all $x \in \mathbb{Z}_{17}^\times$. Is $\theta$ one-to-one? Is $\theta$ onto? If possible, find the inverse function $\theta^{-1}$.

29. For integers $m, n, b$ with $n > 1$, define $f: \mathbb{Z}_n \to \mathbb{Z}_n$ by $f([x]_n) = [mx + b]_n$.

(a) Show that $f$ is a well-defined function.
(b) Prove that $f$ is a one-to-one correspondence if and only if $\gcd(m, n) = 1$.
(c) If $\gcd(m, n) = 1$, find the inverse function $f^{-1}$.

30. Let $f: S \to T$ be a function, and let $A, B$ be subsets of $S$. Prove the following:
   (a) If $A \subseteq B$, then $f(A) \subseteq f(B)$.
   (b) $f(A \cup B) = f(A) \cup f(B)$
   (c) $f(A \cap B) \subseteq f(A) \cap f(B)$

31. Let $f: S \to T$ be a function. Prove that $f$ is a one-to-one function if and only if $f(A \cap B) = f(A) \cap f(B)$ for all subsets $A, B$ of $S$.

32. Let $f: S \to T$ be a function, and let $X, Y$ be subsets of $T$. Prove the following:
   (a) If $X \subseteq Y$, then $f^{-1}(X) \subseteq f^{-1}(Y)$.
   (b) $f^{-1}(X \cup Y) = f^{-1}(X) \cup f^{-1}(Y)$
   (c) $f^{-1}(X \cap Y) = f^{-1}(X) \cap f^{-1}(Y)$

33. Let $A$ be an $n \times n$ matrix with entries in $\mathbb{R}$. Define a linear transformation $L: \mathbb{R}^n \to \mathbb{R}^n$ by $L(x) = Ax$, for all $x \in \mathbb{R}^n$.

(a) Show that $L$ is an invertible function if and only if $\det(A) \neq 0$.
(b) Show that if $L$ is either one-to-one or onto, then it is invertible.

34. Let $A$ be an $m \times n$ matrix with entries in $\mathbb{R}$, and assume that $m > n$. Define a linear transformation $L: \mathbb{R}^n \to \mathbb{R}^m$ by $L(x) = Ax$, for all $x \in \mathbb{R}^n$. Show that $L$ is a one-to-one function if $\det(A^T A) \neq 0$, where $A^T$ is the transpose of $A$.

35. Let $A$ be an $n \times n$ matrix with entries in $\mathbb{R}$. Define a linear transformation $L: \mathbb{R}^n \to \mathbb{R}^n$ by $L(x) = Ax$, for all $x \in \mathbb{R}^n$. Prove that $L$ is one-to-one if and only if no eigenvalue of $A$ is equal to zero.

Note: A vector $x$ is called an eigenvector of $A$ if it is nonzero and there exists a scalar $\lambda$ such that $Ax = \lambda x$, and in this case $\lambda$ is called an eigenvalue of $A$.

MORE PROBLEMS: §2.1

36. In each of the following parts, determine whether the given function is one-to-one and whether it is onto.

†(a) $f: \mathbb{Z}_{12} \to \mathbb{Z}_{12}; f([x]_{12}) = [7x + 3]_{12}$, for all $[x]_{12} \in \mathbb{Z}_{12}$
(b) $f: \mathbb{Z}_{12} \to \mathbb{Z}_{12}; f([x]_{12}) = [8x + 3]_{12}$, for all $[x]_{12} \in \mathbb{Z}_{12}$
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37. For each one-to-one and onto function in Problem 36, find the inverse of the function.

38. Define $f : \mathbb{Z} \to \mathbb{Z}$, by $f([x]_4) = [x]_4^2$, for all $[x]_4 \in \mathbb{Z}$.
   
   (a) Show that $f$ is a well-defined function.
   
   (b) Show that $f$ is one-to-one and onto.

39. Define $f : \mathbb{Z}_{10} \to \mathbb{Z}_{11}$, by $f([m]_{10}) = [2m]_{11}$, for all $[m]_{10} \in \mathbb{Z}_{10}$.
   
   (a) Show that $f$ is a well-defined function.
   
   (b) Is $f$ one-to-one and onto?

40. Define $f : \mathbb{Z}_8 \to \mathbb{Z}_{16}$, by $f([m]_8) = [3m]_{16}$, for all $[m]_8 \in \mathbb{Z}_8$.
   
   (a) Show that $f$ is a well-defined function.
   
   (b) Is $f$ one-to-one and onto?

41. Show that each of the following formulas yields a well-defined function.
   
   (a) $f : \mathbb{Z}_8 \to \mathbb{Z}_8$ defined by $f([x]_8) = [3x^2 - 3x + 1]_8$, for all $[x]_8 \in \mathbb{Z}_8$
   
   (b) $f : \mathbb{Z}_{12} \to \mathbb{Z}_8$ defined by $f([x]_{12}) = [2x^2 - 4x + 6]_8$, for all $[x]_{12} \in \mathbb{Z}_{12}$
   
   (b) $f : \mathbb{Z}_{15} \to \mathbb{Z}_5$ defined by $f([x]_{15}) = [3x^3]_5$, for all $[x]_{15} \in \mathbb{Z}_{15}$

42. Consider the function $f : \mathbb{Z}_{10} \to \mathbb{Z}_{10}$ defined by $f([x]_{10}) = [3x + 4]_{10}$. Show that $f$ is one-to-one and onto by computing all values of the function. Then find a formula of the type $g([x]_{10}) = [mx + b]_{10}$ that gives the inverse of $f$.

43. Let $n$ be a positive integer. Show that $i : \mathbb{Z}_n^\times \to \mathbb{Z}_n^\times$ defined by $i([x]_n) = [x]_n^{-1}$, for all $[x]_n \in \mathbb{Z}_n^\times$ is a well-defined one-to-one correspondence.

44. Let $m, n$ be positive integers with $m \mid n$. Show that $\pi : \mathbb{Z}_n^\times \to \mathbb{Z}_m^\times$ defined by $f([x]_n) = [x]_m$, for all $[x]_n \in \mathbb{Z}_n^\times$ is a well-defined function.

45. Let $m, n$ be positive integers with $m \mid n$, and let $k$ be any integer. Show that $f : \mathbb{Z}_n^\times \to \mathbb{Z}_m^\times$ defined by $f([x]_n) = [x]_m^k$, for all $[x]_n \in \mathbb{Z}_n^\times$ is a well-defined function.

*Hint:* We know that $g : \mathbb{Z}_m^\times \to \mathbb{Z}_m^\times$ defined by $g([x]_m) = [x]_m^k$ is a function if $k$ is a positive integer. To prove the general case if $k$ is negative, you can use the composite function $i \circ g \circ \pi$, where $i$ and $\pi$ are the functions in Problems 43 and 44, respectively.