3.7 Homomorphisms

from A Study Guide for Beginner’s by J.A.Beachy,
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21. Find all group homomorphisms from \( \mathbb{Z}_4 \) into \( \mathbb{Z}_{10} \).

**Solution:** As noted in Example 3.7.7, any group homomorphism from \( \mathbb{Z}_n \) into \( \mathbb{Z}_k \) must have the form \( \phi([x]_n) = [mx]_k \), for all \([x]_n \in \mathbb{Z}_n\). Under any group homomorphism \( \phi : \mathbb{Z}_4 \to \mathbb{Z}_{10} \), the order of \( \phi([1]_4) \) must be a divisor of 4 and of 10, so the only possibilities are \( o(\phi([1]_4)) = 1 \) or \( o(\phi([1]_4)) = 2 \). Thus \( \phi([1]_4) = [0]_{10} \), which defines the zero function, or else \( \phi([1]_4) = [5]_{10} \), which leads to the formula \( \phi([x]_4) = [5x]_{10} \), for all \([x]_4 \in \mathbb{Z}_4\).

22. (a) Find the formulas for all group homomorphisms from \( \mathbb{Z}_{18} \) into \( \mathbb{Z}_{30} \).

**Solution:** As noted in Example 3.7.7, any group homomorphism from \( \mathbb{Z}_{18} \) into \( \mathbb{Z}_{30} \) must have the form \( \phi([x]_{18}) = [mx]_{30} \), for all \([x]_{18} \in \mathbb{Z}_{18}\). Since \gcd(18, 30) = 6, the possible orders of \([m]_{30} = \phi([1]_{18}) \) are 1, 2, 3, 6. The corresponding choices for \([m]_{30} \) are \([0]_{30} \) (order 1), \([15]_{30} \) (order 2), \([10]_{30} \) and \([20]_{30} \) (order 3), and \([5]_{30} \) and \([25]_{30} \) (order 6).

(b) Choose one of the nonzero formulas in part (a), and name it \( \phi \). Find \( \phi(\mathbb{Z}_{18}) \) and \( \ker(\phi) \), and show how elements of \( \phi(\mathbb{Z}_{18}) \) correspond to equivalence classes of \( \sim_\phi \).

**Solution:** For example, consider \( \phi([x]_{18}) = [5x]_{30} \). The image of \( \phi \) consists of the multiples of 5 in \( \mathbb{Z}_{30} \), which are 0, 5, 10, 15, 20, 25. We have \( \ker(\phi) = \{0, 6, 12\} \), and using Proposition 3.7.9 to find the equivalence classes of \( \sim_\phi \), we add 1, 2, 3, 4, and 5, respectively, to the kernel. We have the following correspondence:

\[
{[0]_{18}, [6]_{18}, [12]_{18}} \leftrightarrow \phi([0]_{18}) = [0]_{30}, \quad {[3]_{18}, [9]_{18}, [15]_{18}} \leftrightarrow \phi([3]_{18}) = [15]_{30}, \\
{[1]_{18}, [7]_{18}, [13]_{18}} \leftrightarrow \phi([1]_{18}) = [5]_{30}, \quad {[4]_{18}, [10]_{18}, [16]_{18}} \leftrightarrow \phi([4]_{18}) = [20]_{30}, \\
{[2]_{18}, [8]_{18}, [14]_{18}} \leftrightarrow \phi([2]_{18}) = [10]_{30}, \quad {[5]_{18}, [11]_{18}, [17]_{18}} \leftrightarrow \phi([5]_{18}) = [25]_{30}.
\]

23. (a) Show that \( \mathbb{Z}_{11}^\times \) is cyclic, with generator \([2]_{11}\).

**Solution:** An element of \( \mathbb{Z}_{11}^\times \) can have order 1, 2, 5, or 10. Since \( 2^2 \equiv 4 \neq 1 \) and \( 2^5 \equiv 10 \neq 1 \), it follows that \([2]_{11}\) cannot have order 2 or 5, so it must have order 10.

(b) Show that \( \mathbb{Z}_{19}^\times \) is cyclic, with generator \([2]_{19}\).

**Solution:** Since \( \mathbb{Z}_{19}^\times \) has order 18, the order of \([2]_{19}\) is 2, 3, 6, or 18. The element \([2]_{19}\) is a generator for \( \mathbb{Z}_{19}^\times \) because it has order 18, since \( 2^2 \equiv 4 \neq 1, 2^3 \equiv 8 \neq 1, \) and \( 2^6 \equiv (2^3)^2 \equiv 7 \neq 1 \).

(c) Completely determine all group homomorphisms from \( \mathbb{Z}_{19}^\times \) into \( \mathbb{Z}_{11}^\times \).

**Solution:** Any group homomorphism \( \phi : \mathbb{Z}_{19}^\times \to \mathbb{Z}_{11}^\times \) is determined by its value on the generator \([2]_{19}\), and the order of \( \phi([2]_{19}) \) must be a common divisor of 18 and 10. Thus the only possible order of \( \phi([2]_{19}) \) is 1 or 2, so either \( \phi([2]_{19}) = [1]_{11} \) or \( \phi([2]_{19}) = [0]_{11} = [-1]_{11} \). In the first case, \( \phi([x]_{19}) = [1]_{11} \) for all \([x]_{19} \in \mathbb{Z}_{19}^\times \), and in the second case \( \phi([x]_{19}) = [-1]_{11} \), for all \([x]_{19} = [2]_{19} \in \mathbb{Z}_{19}^\times \).
24. Define \( \phi : \mathbb{Z}_4 \times \mathbb{Z}_6 \to \mathbb{Z}_4 \times \mathbb{Z}_3 \) by \( \phi([x]_4, [y]_6) = ([x + 2y]_4, [y]_3) \).

(a) Show that \( \phi \) is a well-defined group homomorphism.

**Solution:** If \( y_1 \equiv y_2 \pmod{6} \), then \( 2y_1 - 2y_2 \) is divisible by 12, so \( 2y_1 \equiv 2y_2 \pmod{4} \), and then it follows quickly that \( \phi \) is a well-defined function. It is also easy to check that \( \phi \) preserves addition.

(b) Find the kernel and image of \( \phi \), and apply Theorem 3.7.8.

**Solution:** If \( ([x]_4, [y]_6) \) belongs to \( \ker(\phi) \), then \( y \equiv 0 \pmod{3} \), so \( y = 0 \) or \( y = 3 \). If \( y = 0 \), then \( x = 0 \), and if \( y = 3 \), then \( x = 2 \). Thus the elements of the kernel \( K \) are \(([0]_4, [0]_6)\) and \(([2]_4, [3]_6)\).

It follows that there are \( 24/2 = 12 \) equivalence classes determined by \( \phi \). These are in one-to-one correspondence with the elements of the image, so \( \phi \) must map \( \mathbb{Z}_4 \times \mathbb{Z}_6 \) onto \( \mathbb{Z}_4 \times \mathbb{Z}_3 \). Thus \( (\mathbb{Z}_4 \times \mathbb{Z}_6)/\phi \cong \mathbb{Z}_4 \times \mathbb{Z}_3 \).

25. Let \( n \) and \( m \) be positive integers, such that \( m \) is a divisor of \( n \). Show that \( \phi : \mathbb{Z}_n^\times \to \mathbb{Z}_m^\times \) defined by \( \phi([x]_n) = [x]_m \), for all \( [x]_n \in \mathbb{Z}_n^\times \), is a well-defined group homomorphism.

**Solution:** First, note that \( \phi \) is a well-defined function, since if \( [x_1]_n = [x_2]_n \) in \( \mathbb{Z}_n^\times \), then \( n \mid (x_1 - x_2) \), and this implies that \( m \mid (x_1 - x_2) \), since \( m \mid n \). Thus \( [x_1]_m = [x_2]_m \), and so \( \phi([x_1]_n) = \phi([x_2]_n) \).

Next, \( \phi \) is a homomorphism since for \( [a]_n, [b]_n \in \mathbb{Z}_n^\times \), we have \( \phi([a]_n [b]_n) = \phi([ab]_n) = [ab]_m = [a]_m [b]_m = \phi([a]_n) \phi([b]_n) \).

26. For the group homomorphism \( \phi : \mathbb{Z}_{36}^\times \to \mathbb{Z}_{12}^\times \) defined by \( \phi([x]_{36}) = [x]_{12} \), for all \( [x]_{36} \in \mathbb{Z}_{36}^\times \), find the kernel and image of \( \phi \), and apply Theorem 3.7.8.

**Solution:** The previous problem shows that \( \phi \) is a group homomorphism. It is evident that \( \phi \) maps \( \mathbb{Z}_{36}^\times \) onto \( \mathbb{Z}_{12}^\times \), since if \( \gcd(x, 12) = 1 \), then \( \gcd(x, 36) = 1 \). The kernel of \( \phi \) consists of the elements in \( \mathbb{Z}_{36}^\times \) that are congruent to 1 mod 12, namely \( [1]_{36}, [13]_{36}, [25]_{36} \). It follows that \( \mathbb{Z}_{36}^\times / \phi \cong \mathbb{Z}_{12}^\times \).

27. Prove that \( \text{SL}_n(\mathbb{R}) \) is a normal subgroup of \( \text{GL}_n(\mathbb{R}) \).

**Solution:** Let \( G = \text{GL}_n(\mathbb{R}) \) and \( N = \text{SL}_n(\mathbb{R}) \). The condition we need to check, that \( g x g^{-1} \subseteq N \) for all \( n \in N \) and all \( g \in G \), translates into the condition that if \( P \) is any invertible matrix and \( \det(A) = 1 \), then \( P A P^{-1} \) has determinant 1. This follows immediately from the fact that

\[
\det(P A P^{-1}) = \det(P) \det(A) \det(P^{-1}) = \det(P) \det(A) \frac{1}{\det(P)} = \det(A),
\]

which you may remember from your linear algebra course as the proposition that similar matrices have the same determinant.

**Alternate solution:** Here is a slightly more sophisticated proof. Note that \( \text{SL}_n(\mathbb{R}) \) is the kernel of the determinant homomorphism from \( \text{GL}_n(\mathbb{R}) \) into \( \mathbb{R} \) (Example 3.7.1 in the text shows that the determinant defines a group homomorphism.) The result then follows immediately from Proposition 3.7.4 (a), which shows that the kernel of any group homomorphism is a normal subgroup.
ANSWERS AND HINTS

28. Prove or disprove each of the following assertions:
   (a) The set of all nonzero scalar matrices is a normal subgroup of GL_n(R).
   Answer: This set is the center of GL_n(R), and so it is a normal subgroup.
   (b) The set of all diagonal matrices with nonzero determinant is a normal subgroup of GL_n(R).
   Answer: This set is a subgroup, but it is not normal.

31. Define φ : Z_15^× → Z_15^× by φ([x]_{15}) = [x]^3_{15}, for all [x]_{15} ∈ Z_15^×. Find the kernel and image of φ.
   Answer: The function φ is an isomorphism.

32. Define φ : Z_5^× → Z_5^× by φ([x]_{15}) = [x]^3_{5}. Show that φ is a group homomorphism, and find its kernel and image.
   Answer: The function φ is onto with ker(φ) = {[1]_{15}, [11]_{15}}.

34. How many homomorphisms are there from Z_{12} into Z_4 × Z_3?
   Answer: There are 12, since [1]_{12} can be mapped to any element of Z_4 × Z_3.

36. Define φ : R → C^× by setting φ(x) = e^{ix}, for all x ∈ R. Show that φ is a group homomorphism, and find ker(φ) and the image φ(R).
   Answer: The image is the unit circle in the complex plane, and ker(φ) = {2kπ | k ∈ Z}.

37. Let G be a group, with a subgroup H ⊆ G. Define N(H) = \{g ∈ G | gHg^{-1} = H\}.
   (b) Let G = S_4. Find N(H) for the subgroup H generated by (1, 2, 3) and (1, 2).
   Answer: In this example, N(H) = H.

38. Find all normal subgroups of A_4.
   Answer: \{(1)\}, N = \{(1), (1, 2)(3, 4), (1, 3)(2, 4), (1, 4)(2, 3)\}, and A_4 are the normal subgroups of A_4.