Section 4.4 returns to the setting of high school algebra. The most important theorems here are Eisenstein’s irreducibility criterion for polynomials with integer coefficients (Theorem 4.4.6) and the fundamental theorem of algebra (Theorem 4.4.9), which states that any polynomial of positive degree in \( \mathbb{C}[x] \) has a root in \( \mathbb{C} \). Unfortunately, it is relatively hard to give a strictly algebraic proof of the fundamental theorem, so the proof has to be postponed to Chapter 8 after we have studied some Galois theory.

**SOLVED PROBLEMS: §4.4**

21. Factor \( x^5 - 10x^4 + 24x^3 + 9x^2 - 33x - 12 \) over \( \mathbb{Q} \).

22. Show that \( x^3 + (3m - 1)x + (3n + 1) \) is irreducible in \( \mathbb{Q}[x] \) for all \( m, n \in \mathbb{Z} \).

23. Use Eisenstein’s criterion to show that \( x^4 + 120x^3 - 90x + 60 \) is irreducible over \( \mathbb{Q} \).

24. Factor \( x^8 - 1 \) over \( \mathbb{C} \).

25. Factor \( x^4 - 2 \) over \( \mathbb{C} \).

26. Factor \( x^3 - 2 \) over \( \mathbb{C} \).

**MORE PROBLEMS: §4.4**

27.† Find all rational roots of \( f(x) = x^3 - x^2 + x - 1 \).

28.† Find all rational roots of \( g(x) = x^5 - 4x^4 - 2x^3 + 14x^2 - 3x + 18 \).

29.† Factor the polynomials \( f(x) \) and \( g(x) \) in Problems 27 and 28 and use the factorizations to find \( \gcd(f(x), g(x)) \) in \( \mathbb{Q}[x] \).

30.† Find all rational roots of \( f(x) = x^5 - 8x^4 + 25x^3 - 38x^2 + 28x - 8 \).

31.† Find all rational roots of \( g(x) = x^5 - x^4 - 2x^3 + 2x^2 + x - 1 \).

32.† Factor the polynomials \( f(x) \) and \( g(x) \) in Problems 30 and 31 and use the factorizations to find \( \gcd(f(x), g(x)) \) in \( \mathbb{Q}[x] \).

33. Find all rational roots of \( x^5 - 6x^4 + 3x^3 - 3x^2 + 2x - 12 \).

34. Use Eisenstein’s criterion to show that \( x^4 - 10x^2 + 1 \) is irreducible over \( \mathbb{Q} \).

35.† Show that \( x^4 - 4x^3 + 13x^2 - 32x + 43 \) is irreducible over \( \mathbb{Q} \). Use Eisenstein’s criterion.

36.† Show that \( x^5 + 5x^4 - 40x^2 - 75x - 48 \) is irreducible over \( \mathbb{Q} \). Use Eisenstein’s criterion.
37. Show that \( p(x) = [(x-1)(x-2) \cdots (x-n)] - 1 \) is irreducible over \( \mathbb{Q} \) for all integers \( n > 1 \).

38. Show that \( x^4 - 3x^3 + 2x^2 + x + 5 \) has \( -2 - i \) as a root in \( \mathbb{C} \). Then find the other roots of \( f(x) \) in \( \mathbb{C} \).

39. Factor \( x^6 - 1 \) over \( \mathbb{C} \).

The following subset of \( M_2(\mathbb{C}) \) is called the set of **quaternions**.

\[
H = \left\{ \begin{bmatrix} z & w \\ -\overline{w} & \overline{z} \end{bmatrix} \right\} | z, w \in \mathbb{C}
\]

If we represent the complex numbers \( z \) and \( w \) as \( z = a + bi \) and \( w = c + di \), then

\[
\begin{bmatrix} z & w \\ -\overline{w} & \overline{z} \end{bmatrix} = a \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + b \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix} + c \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} + d \begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix}.
\]

If we let

\[
1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad i = \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}, \quad j = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \quad k = \begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix},
\]

then we can write

\[
H = \left\{ a \cdot 1 + b i + c j + d k \mid a, b, c, d \in \mathbb{R} \right\}.
\]

Direct computations with the elements \( i, j, k \) show that we have the following identities:

\[
i^2 = j^2 = k^2 = -1;
\]

\[
ij = k, \quad jk = i, \quad ki = j; \quad ji = -k, \quad kj = -i, \quad ik = -j.
\]

These identities show that \( H \) is closed under matrix addition and multiplication. It can be shown that \( H \) satisfies all of the axioms of a field, with the exception of the commutative law for multiplication.

The determinant of the matrix corresponding to \( a \cdot 1 + b i + c j + d k \) is \( z\overline{z} + w\overline{w} = a^2 + b^2 + c^2 + d^2 \), and this observation shows that each nonzero element of \( H \) has a multiplicative inverse. The full name for \( H \) is the **division ring of real quaternions**. The notation \( H \) is used in honor of Hamilton, who discovered the quaternions after about ten years of trying to construct a field using 3-tuples of real numbers. He finally realized that if he would sacrifice the commutative law and extend the multiplication to 4-tuples then he could construct a division ring.

40.† Show that there are infinitely many quaternions that are roots of the polynomial \( x^2 + 1 \).