5.3 Ideals and Factor Rings

from A Study Guide for Beginner’s by J.A.Beachy,
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This section considers factor rings, using results on factor groups. The analogy to the material in Section 3.8 is a very close one, and the first goal is to determine which subsets of a ring will play the same role as that of normal subgroups of a group. These are the ideals of the ring, which correspond to the kernels of ring homomorphisms from the ring to other rings. (Remember that normal subgroups are precisely the kernels of group homomorphisms.)

The most important examples of principal ideal domains are the ring \( \mathbb{Z} \), in which any nonzero ideal \( I \) is generated by the smallest positive integer in \( I \), and the ring \( F[x] \) of polynomials over a field \( F \), in which any nonzero ideal \( I \) is generated by the monic polynomial of minimal degree in \( I \).

Let \( I \) and \( J \) be ideals of the commutative ring \( R \). Exercise 5.3.13 shows that the sum of \( I \) and \( J \), denoted

\[
I + J = \{ x \in R \mid x = a + b \text{ for some } a \in I, b \in J \}
\]

is an ideal of \( R \). Exercise 5.3.14 defines the product of the ideals \( I \) and \( J \), which is a bit more complicated:

\[
IJ = \{ \sum_{i=1}^{n} a_i b_i \mid a_i \in I, b_i \in J, n \in \mathbb{Z}^+ \}
\]

The product \( IJ \) is also an ideal of \( R \). Exercise 5.3.11 defines the annihilator of \( a \in R \) to be \( \text{Ann}(a) = \{ x \in R \mid xa = 0 \} \).

SOLVED PROBLEMS: §5.3

27. Give an example to show that the set of all zero divisors of a commutative ring need not be an ideal of the ring.

28. Show that in \( R[x] \) the set of polynomials whose graph passes through the origin and is tangent to the x-axis at the origin is an ideal of \( R[x] \).

29. To illustrate Proposition 5.3.7 (b), give the lattice diagram of ideals of \( \mathbb{Z}_{100} = \mathbb{Z}/\langle 100 \rangle \), and the lattice diagram of ideals of \( \mathbb{Z} \) that contain \( \langle 100 \rangle \).

30. Let \( R \) be the ring \( \mathbb{Z}_2[x]/\langle x^3 + 1 \rangle \).
   (a) Find all ideals of \( R \).
   (b) Find the units of \( R \).
   (c) Find the idempotent elements of \( R \).

31. Let \( S \) be the ring \( \mathbb{Z}_2[x]/\langle x^3 + x \rangle \).
   (a) Find all ideals of \( S \).
   (b) Find the units of \( S \).
   (c) Find the idempotent elements of \( S \).
32. Show that the rings $R$ and $S$ in Problems 30 and 31 are isomorphic as abelian groups, but not as rings.

33. Let $I, J$ be ideals of the commutative ring $R$, and for $r \in R$, define the function $\phi : R \rightarrow R/I \oplus R/J$ by $\phi(r) = (r + I, r + J)$.
   (a) Show that $\phi$ is a ring homomorphism, with $\ker(\phi) = I \cap J$.
   (b) Show that if $I + J = R$, then $\phi$ is onto, and thus $R/(I \cap J) \cong R/I \oplus R/J$.

34. Let $I, J$ be ideals of the commutative ring $R$. Show that if $I + J = R$, then $I^2 + J^2 = R$.

35. Show that $\langle x^2 + 1 \rangle$ is a maximal ideal of $R[x]$.

36. Is $\langle x^2 + 1 \rangle$ a maximal ideal of $\mathbb{Z}_2[x]$?

37. Let $R$ and $S$ be commutative rings, and let $\phi : R \rightarrow S$ be a ring homomorphism.
   (a) Show that if $I$ is an ideal of $S$, then $\phi^{-1}(I) = \{a \in R \mid \phi(a) \in I\}$ is an ideal of $R$.
   (b) Show that if $P$ is a prime ideal of $S$, then $\phi^{-1}(P)$ is a prime ideal of $R$.

38. Prove that in a Boolean ring every prime ideal is maximal.

39. In $R = \mathbb{Z}[i]$, let $I = \{m + ni \mid m \equiv n \pmod{2}\}$.
   (a) Show that $I$ is an ideal of $R$.
   (b) Find a familiar commutative ring isomorphic to $R/I$.

MORE PROBLEMS: §5.3

40. To illustrate Proposition 3.5.7 (b), give the lattice diagram of ideals of $\mathbb{Z}_{45}$ and the lattice diagram of ideals of $\mathbb{Z}$ that contain $\langle 45 \rangle$.

41. Let $R$ and $S$ be commutative rings, and let $\phi : R \rightarrow S$ be a ring homomorphism. Show that if $\phi$ is onto and every ideal of $R$ is a principal ideal, then the same condition holds in $S$.

42. Show that if $I, J$ are ideals of the commutative ring $R$ with $I + J = R$, then $I \cap J = IJ$.

43. Let $I, J$ be ideals of the commutative ring $R$. Show that $\{r \in R \mid rx \in J \text{ for all } x \in I\}$ is an ideal of $R$.

44. Prove that $R/\{0\}$ is isomorphic to $R$, for any commutative ring $R$.

45.† Let $P$ and $Q$ be maximal ideals of the commutative ring $R$. Show that $R/(P \cap Q) \cong R/P \oplus R/Q$.

46. Suppose that $\{I_n\}_{n \in \mathbb{N}}$ is a set of ideals of the commutative ring $R$ such that $I_n \subset I_m$ whenever $n \leq m$. Prove that $\cup_{n \in \mathbb{N}} I_n$ is an ideal of $R$. 
47. Let $R$ be the set of all $3 \times 3$ matrices of the form
\[
\begin{bmatrix}
a & 0 & 0 \\
b & a & 0 \\
c & b & a
\end{bmatrix}
\] with $a, b, c \in \mathbb{Z}$. (See Problem 5.1.27.) Let $I$ be the subset of $R$ consisting of all matrices with $a = 0$. Show that $I^2 \neq (0)$ but $I^3 = (0)$.

48.† Show that in $\mathbb{Z}[\sqrt{2}]$ the principal ideal generated by $\sqrt{2}$ is a maximal ideal.

49. In $R = \mathbb{Z}[i]$, let $I$ be the principal ideal generated by 5. Prove that $R/I \cong \mathbb{Z}_5 \oplus \mathbb{Z}_5$.

50. Let $R$ be a commutative ring, and let $I, J, K$ be ideals of $R$. Prove the following facts.

(a) $(IJ)K = I(JK)$
(b) $(I \cdot (J \cap K) \subseteq IJ \cap IK$
(c) $I \cap (J + K) \supseteq (I \cap J) + (I \cap K)$
(d) $I + (J \cap K) \subset (I + J) \cap (I + K)$