5.4 Quotient Fields

from A Study Guide for Beginner’s by J.A.Beachy,
a supplement to Abstract Algebra by Beachy / Blair

We know that any subring of a field is an integral domain. This section contains a converse: any integral domain is isomorphic to a subring of a field. The proof constructs a field that is as small as possible, and this field contains no more elements than absolutely necessary to provide inverses for the elements of the domain. The example to keep in mind is that of the ring $\mathbb{Z}$ thought of as a subring of the field $\mathbb{Q}$.

SOLVED PROBLEMS: §5.4

15. Let $F$ be a field. Explain why $Q(F)$ is isomorphic to $F$. Why can’t we just say that $Q(F) = F$?

16. Find the quotient field of $\mathbb{Z}_2[x]$.

17. Prove that if $D_1$ and $D_2$ are isomorphic integral domains, then $Q(D_1) \cong Q(D_2)$.

MORE PROBLEMS: §5.4

18.† Find the quotient field of $\mathbb{Z}[\sqrt{3}i]$.

19.† Find the quotient field of $D = \left\{ \begin{bmatrix} m & n \\ -3n & m \end{bmatrix} \middle| m, n \in \mathbb{Z} \right\}$.

20.† Find the quotient field of $D = \left\{ \begin{bmatrix} m & n \\ -n & m \end{bmatrix} \middle| m, n \in \mathbb{Z} \right\}$.

21. Let $R$ be a set that satisfies all properties of a commutative ring, with the exception of the existence of an identity element $1$. Show that if $R$ has no nonzero divisors of zero, then it has a quotient field (which must necessarily contain an identity element).

22. Let $R$ be a commutative ring. A nonempty subset $S$ of $R$ is called a multiplicative set if $ab \in S$ for all $a, b \in S$, and $0 \notin S$.

(a) Let $S$ be a multiplicative set of $R$. Show that the relation defined on $R \times S$ by $(a, c) \sim (b, d)$ if $s(ad - bc) = 0$ for some $s \in S$ is an equivalence relation.

(b) Denote the equivalence class of $(a, c) \in R \times S$ by $[a, c]$, and denote the set of equivalence classes by $R_S$. Show that the following addition and multiplication of equivalence classes is well-defined, for $a, b \in R$ and $c, d \in S$.

$$[a, c] + [b, d] = [ad + bc, cd] \quad \text{and} \quad [a, c] \cdot [b, d] = [ab, dc].$$

(c) Show that $R_S$ is a commutative ring under the above operations.