Chapter 5 Review Problems

from A Study Guide for Beginner’s by J.A.Beachy,
a supplement to Abstract Algebra by Beachy / Blair

1. Let $R$ be the ring with 8 elements consisting of all $3 \times 3$ matrices with entries in $\mathbb{Z}_2$ which have the following form:

$$
\begin{bmatrix}
    a & 0 & 0 \\
    0 & a & 0 \\
    b & c & a
\end{bmatrix}
$$

You may assume that the standard laws for addition and multiplication of matrices are valid.

(a) Show that $R$ is a commutative ring (you only need to check closure, the existence of a 1, and commutativity of multiplication).

(b) Find all units of $R$, and all nilpotent elements of $R$.

(c) Find all idempotent elements of $R$.

2. Let $R$ be the ring $\mathbb{Z}_2[x]/\langle x^2 + 1 \rangle$. Show that although $R$ has 4 elements, it is not ring isomorphic to either $\mathbb{Z}_4$ or $\mathbb{Z}_2 \oplus \mathbb{Z}_2$.

3. Let $R$ and $S$ be commutative rings. Prove that $R \oplus S \cong S \oplus R$.

4. For the element $a = (0, 2)$ of the ring $R = \mathbb{Z}_{12} \oplus \mathbb{Z}_8$, find $\text{Ann}(a) = \{ r \in R \mid ra = 0 \}$.

Check that $\text{Ann}(a)$ is an ideal of $R$.

5. Let $R$ be the ring $\mathbb{Z}_2[x]/\langle x^4 + 1 \rangle$, and let $I$ be the set of all congruence classes in $R$ of the form $[f(x)(x^2 + 1)]$.

(a) Show that $I$ is an ideal of $R$.

(b) Show that $R/I \cong \mathbb{Z}_2[x]/\langle x^2 + 1 \rangle$.

(c) Is $I$ a prime ideal of $R$?

Hint: If you use the fundamental homomorphism theorem, you can do the first two parts together.

6. Find all maximal ideals, and all prime ideals, of $\mathbb{Z}_{36} = \mathbb{Z}/36\mathbb{Z}$.

7. Let $I$ be the subset of $\mathbb{Z}[x]$ consisting of all polynomials with even coefficients. Prove that $I$ is a prime ideal; prove that $I$ is not maximal.

8. Let $\mathbb{Z}[i]$ be the ring of Gaussian integers, i.e. the subring of $\mathbb{C}$ given by

$$
\mathbb{Z}[i] = \{ m + ni \in \mathbb{C} \mid m, n \in \mathbb{Z} \}.
$$

(a) Define $\phi : \mathbb{Z}[i] \rightarrow \mathbb{Z}_2$ by $\phi(m + ni) = [m + n]_2$. Prove that $\phi$ is a ring homomorphism. Find $\ker(\phi)$ and show that it is a principal ideal of $\mathbb{Z}[i]$. 

(b) For any prime number $p$, define $\theta : \mathbb{Z}[i] \rightarrow \mathbb{Z}_p[x]/\langle x^2 + 1 \rangle$ by $\theta(m+ni) = [m+nx]$. Prove that $\theta$ is an onto ring homomorphism.