6.1 Algebraic Elements

In this section we often start with a known field \(K\), and then construct a larger field \(F\). Recall Definition 4.3.1: the field \(F\) is said to be an extension field of the field \(K\) if \(K\) is a subset of \(F\) which is a field under the operations of \(F\). Equivalently, \(K\) is a subfield of \(F\), often called the base field in this situation. If \(u \in F\) is a root of some nonzero polynomial \(f(x) \in K[x]\), and \(p(x)\) has minimal degree among all polynomials of which \(u\) is a root, then we can use the division algorithm to show that \(p(x) | f(x)\). It is very useful to know that the field \(K(u)\) is isomorphic to \(K[x]/ \langle p(x) \rangle\), so that calculations can be done in either field.

SOLVED PROBLEMS: §6.1

13. Let \(u\) be a root of the polynomial \(x^3 + 3x + 3\). In \(Q(u)\), express \((7 - 2u + u^2)^{-1}\) in the form \(a + bu + cu^2\).

14. Find the minimal polynomial of the real number \(1 + \sqrt{2}\) over \(Q\).

15. Find the minimal polynomial of the complex number \(1 + \sqrt{3}i\) over \(Q\).

16. (a) Show that \(Q(\sqrt{2} + i) = Q(\sqrt{2}, i)\).

(b) Find the minimal polynomial of the complex number \(\sqrt{2} + i\) over \(Q\).

17. Show that the polynomial \(f(x) = x^2 + x - 1\) is irreducible over the field \(K = \mathbb{Z}_3\), but has two roots in the extension field \(F = \mathbb{Z}_3[x]/ \langle x^2 + 1 \rangle\).

18. Let \(F\) be an extension field of \(K\). Let \(G\) be the set of all automorphisms \(\phi : F \to F\) such that \(\phi(x) = x\) for all \(x \in K\). Show that \(G\) is a group (under composition of functions).

MORE PROBLEMS: §6.1

19. Find the minimal polynomial of the complex number \(\sqrt{2} + \sqrt{3}i\) over \(Q\).

20. Find \([Q(\sqrt{2}, \sqrt{3}) : Q(\sqrt{6})]\).

21. Let \(a + bi\) be a complex number that is algebraic over \(Q\). Show that \(\sqrt{a + bi}\) is algebraic over \(Q\), and find its minimal polynomial over \(Q\) in terms of the minimal polynomial of \(a + bi\) over \(Q\).

22. Let \(F\) be an extension field of \(K\), and let \(u \in F\). Show that if \(f(u)\) is algebraic over \(K\) for some \(f(x) \in K[x]\), then \(u\) itself is algebraic over \(K\).

23. Let \(a\) and \(b\) be nonzero rational numbers. Show that \(Q(\sqrt{a}) = Q(\sqrt{b})\) if and only if there exists \(c \in Q\) with \(a = bc^2\).

24. Let \(F\) be an extension field of \(K\), and let \(u, v \in F\). Prove that \(K(u, v) = K(v, u)\).

25. Show that if \(u\) and \(v\) are transcendental over \(Q\), then either \(uv\) or \(u + v\) is transcendental over \(Q\).