2.1 Functions

Besides reading Section 2.1, it might help to get out your calculus textbook and review composite functions, one-to-one and onto functions, and inverse functions. The functions $f : \mathbf{R} \to \mathbf{R}^+$ and $g : \mathbf{R}^+ \to \mathbf{R}$ defined by $f(x) = e^x$, for all $x \in \mathbf{R}$, and $g(y) = \ln y$, for all $y \in \mathbf{R}^+$, provide one of the most important examples of a pair of inverse functions.

Definition 2.1.1, the definition of function, is stated rather formally in terms of ordered pairs. (Think of this as a definition given in terms of the "graph" of the function.) In terms of actually using this definition, the text almost immediately goes back to what might be a more familiar definition: a function $f : S \to T$ is a "rule" that assigns to each element of S a unique element of T.

One of the most fundamental ideas of abstract algebra is that algebraic structures should be thought of as essentially the same if the only difference between them is the way elements have been named. To make this precise we will say that structures are the same if we can set up an invertible function from one to the other that preserves the essential algebraic structure. That makes it especially important to understand the concept of an inverse function, as introduced in this section.

SOLVED PROBLEMS: §2.1

20. The "Vertical Line Test" from calculus says that a curve in the xy-plane is the graph of a function of x if and only if no vertical line intersects the curve more than once. Explain why this agrees with Definition 2.1.1.

21. The "Horizontal Line Test" from calculus says that a function is one-to-one if and only if no horizontal line intersects its graph more than once. Explain why this agrees with Definition 2.1.4.

    more than one

22. In calculus the graph of an inverse function f^{-1} is obtained by reflecting the graph of f about the line $y = x$. Explain why this agrees with Definition 2.1.7.

23. Let A be an $n \times n$ matrix with entries in $\mathbf{R}$. Define a linear transformation $L : \mathbf{R}^n \to \mathbf{R}^n$ by $L(\mathbf{x}) = \mathbf{A}\mathbf{x}$, for all $\mathbf{x} \in \mathbf{R^n}$.

    (a) Show that L is an invertible function if and only if $\det(A) \neq 0$.

    (b) Show that if L is either one-to-one or onto, then it is invertible.

24. Let A be an $m \times n$ matrix with entries in $\mathbf{R}$, and assume that $m > n$. Define a linear transformation $L : \mathbf{R}^n \to \mathbf{R}^m$ by $L(\mathbf{x}) = \mathbf{A}\mathbf{x}$, for all $\mathbf{x} \in \mathbf{R^n}$. Show that L is a one-to-one function if $\det(A^T A) \neq 0$, where A^T is the transpose of A.

25. Let A be an $n \times n$ matrix with entries in $\mathbf{R}$. Define a linear transformation $L : \mathbf{R}^n \to \mathbf{R}^n$ by $L(\mathbf{x}) = \mathbf{A}\mathbf{x}$, for all $\mathbf{x} \in \mathbf{R^n}$. Prove that L is one-to-one if and only if no eigenvalue of A is zero.

    Note: A vector $\mathbf{x}$ is called an eigenvector of A if it is nonzero and there exists a scalar λ such a that $A\mathbf{x} = \lambda\mathbf{x}$.

26. Let a be a fixed element of $\mathbf{Z}_{17}^\times$. Define the function $\theta : \mathbf{Z}_{17}^\times \to \mathbf{Z}_{17}^\times$ by $\theta(x) = ax$, for all $x \in \mathbf{Z}_{17}^\times$. Is θ one to one? Is θ onto? If possible, find the inverse function θ^{-1}.