2.2 Equivalence Relations

In a variety of situations it is useful to split a set up into subsets in which the elements have some property in common. You are already familiar with one of the important examples: in Chapter 1 we split the set of integers up into subsets, depending on the remainder when the integer is divided by the fixed integer n. This led to the concept of congruence modulo n, which is a model for our general notion of an *equivalence relation*.

In this section you will find three different points of view, looking at the one idea of splitting up a set S from three distinct vantage points. First there is the definition of an equivalence relation on S, which tells you when two different elements of S belong to the same subset. Then there is the notion of a partition of S, which places the emphasis on describing the subsets. Finally, it turns out that every partition (and equivalence relation) really comes from a function $f : S \to T$, where we say that x_1 and x_2 are equivalent if $f(x_1) = f(x_2)$.

The reason for considering several different point of view is that in a given situation one point of view may be more useful than another. Your goal should be to learn about each point of view, so that you can easily switch from one to the other, which is a big help in deciding which point of view to take.

SOLVED PROBLEMS: §2.2

14. On the set $\{(a, b)\}$ of all ordered pairs of positive integers, define $(x_1, y_1) \sim (x_2, y_2)$ if $x_1y_2 = x_2y_1$. Show that this defines an equivalence relation.

15. On the set $\mathbf{C}$ of complex numbers, define $z_1 \sim z_2$ if $||z_1|| = ||z_2||$. Show that $\sim$ is an equivalence relation.

16. Let $\mathbf{u}$ be a fixed vector in $\mathbf{R}^3$, and assume that $\mathbf{u}$ has length 1. For vectors $\mathbf{v}$ and $\mathbf{w}$, define $\mathbf{v} \sim \mathbf{w}$ if $\mathbf{v}\cdot\mathbf{u} = \mathbf{w}\cdot\mathbf{u}$, where $\cdot$ denotes the standard dot product. Show that $\sim$ is an equivalence relation, and give a geometric description of the equivalence classes of $\sim$.

17. For the function $f : \mathbf{R} \to \mathbf{R}$ defined by $f(x) = x^2$, for all $x \in \mathbf{R}$, describe the equivalence relation on $\mathbf{R}$ that is determined by f.

18. For the linear transformation $L : \mathbf{R}^3 \to \mathbf{R}^3$ defined by

$$L(x, y, z) = (x + y + z, x + y + z, x + y + z) ,$$

for all $(x, y, z) \in \mathbf{R}^3$, give a geometric description of the partition of $\mathbf{R}^3$ that is determined by L.

19. Define the formula $f : \mathbf{Z}_{12} \to \mathbf{Z}_{12}$ by $f([x]_{12}) = [x]_{12}^2$, for all $[x]_{12} \in \mathbf{Z}_{12}$. Show that the formula f defines a function. Find the image of f and the set $\mathbf{Z}_{12}/f$ of equivalence classes determined by f.

20. On the set of all $n \times n$ matrices over $\mathbf{R}$, define $A \sim B$ if there exists an invertible matrix P such that $PAP^{-1} = B$. Check that $\sim$ defines an equivalence relation.