2.3 Permutations

This section introduces and studies the last major example that we need before we begin studying groups in Chapter 3. You need to do enough computations so that you will feel comfortable in dealing with permutations.

If you are reading another book along with **Abstract Algebra**, you need to be aware that some authors multiply permutations by reading from left to right, instead of the way we have defined multiplication. Our point of view is that permutations are functions, and we write functions on the left, just as in calculus, so we have to do the computations from right to left.

In the text we noted that if S is any set, and $\mathrm{Sym}(S)$ is the set of all permutations on S, then we have the following properties. (i) If $\sigma, \tau \in \mathrm{Sym}(S)$, then $\tau\sigma \in \mathrm{Sym}(S)$; (ii) $1_S \in \mathrm{Sym}(S)$; (iii) if $\sigma \in \mathrm{Sym}(S)$, then $\sigma^{-1} \in \mathrm{Sym}(S)$. In two of the problems, we need the following definition.

If G is a nonempty subset of $\mathrm{Sym}(S)$, we will say that G is a *group of permutations* if the following conditions hold.

(i) If $\sigma, \tau \in G$, then $\tau\sigma \in G$;

(ii) $1_S \in G$;

(iii) if $\sigma \in G$, then $\sigma^{-1} \in G$.

We will see later that this agrees with Definition 3.6.1 of the text.

SOLVED PROBLEMS: §2.3

13. For the permutation $\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 7 & 5 & 6 & 9 & 2 & 4 & 8 & 1 & 3 \end{pmatrix}$, write σ as a product of disjoint cycles. What is the order of σ? Is σ an even permutation? Compute σ^{-1}.

14. For the permutations $\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 2 & 5 & 1 & 8 & 3 & 6 & 4 & 7 & 9 \end{pmatrix}$ and $\tau = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 1 & 5 & 4 & 7 & 2 & 6 & 8 & 9 & 3 \end{pmatrix}$, write each of these permutations as a product of disjoint cycles: σ, τ, $\sigma\tau$, $\sigma\tau\sigma^{-1}$, σ^{-1}, τ^{-1}, $\tau\sigma$, $\tau\sigma\tau^{-1}$.

15. Let $\sigma = (2,4,9,7,)(6,4,2,5,9)(1,6)(3,8,6) \in S_9$. Write σ as a product of disjoint cycles. What is the order of σ? Compute σ^{-1}.

16. Compute the order of $\tau = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 \\ 7 & 2 & 11 & 4 & 6 & 8 & 9 & 10 & 1 & 3 & 5 \end{pmatrix}$. For $\sigma = (3,8,7)$, compute the order of $\sigma\tau\sigma^{-1}$.

17. Prove that if $\tau \in \mathcal{S}_n$ is a permutation with order m, then $\sigma\tau\sigma^{-1}$ has order m, for any permutation $\sigma \in \mathcal{S}_n$.

18. Show that S_{10} has elements of order 10, 12, and 14, but not 11 or 13.

19. Let S be a set, and let X be a subset of S. Let $G = \{\sigma \in \mathrm{Sym}(S) \mid \sigma(X) \subset X\}$. Prove that G is a group of permutations.

20. Let G be a group of permutations, with $G \subseteq \mathrm{Sym}(S)$, for the set S. Let τ be a fixed permutation in $\mathrm{Sym}(S)$. Prove that

$$\tau G \tau^{-1} = \{\sigma \in \mathrm{Sym}(S) \mid \sigma = \tau\gamma\tau \text{ for some } \gamma \in G\}$$

is a group of permutations.