

3.2 Subgroups

Many times a group is defined by looking at a subset of a known group. If the subset is a group in its own right, using the same operation as the larger set, then it is called a *subgroup*. For instance, any group of permutations is a subgroup of $\text{Sym}(S)$, for some set S . Any group of $n \times n$ matrices (with entries in \mathbf{R}) is a subgroup of $\text{GL}_n(\mathbf{R})$.

If the idea of a subgroup reminds you of studying subspaces in your linear algebra course, you are right. If you only look at the operation of addition in a vector space, it forms an abelian group, and any subspace is automatically a subgroup. Now might be a good time to pick up your linear algebra text and review vector spaces and subspaces.

Lagrange's theorem is very important. It states that in a finite group the number of elements in any subgroup must be a divisor of the total number of elements in the group. This is a useful fact to know when you are looking for subgroups in a given group.

It is also important to remember that every element a in a group defines a subgroup $\langle a \rangle$, consisting of all powers (positive and negative) of the element. This subgroup has $o(a)$ elements, where $o(a)$ is the order of a . If the group is finite, then you only need to look at positive powers, since in that case the inverse a^{-1} of any element can be expressed in the form a^n , for some $n > 0$.

SOLVED PROBLEMS: §3.2

23. Find all cyclic subgroups of \mathbf{Z}_{24}^\times .
24. In \mathbf{Z}_{20}^\times , find two subgroups of order 4, one that is cyclic and one that is not cyclic.
25. (a) Find the cyclic subgroup of S_7 generated by the element $(1, 2, 3)(5, 7)$.
(b) Find a subgroup of S_7 that contains 12 elements. You do not have to list all of the elements if you can explain why there must be 12, and why they must form a subgroup.
26. In $G = \mathbf{Z}_{21}^\times$, show that

$$H = \{[x]_{21} \mid x \equiv 1 \pmod{3}\} \quad \text{and} \quad K = \{[x]_{21} \mid x \equiv 1 \pmod{7}\}$$
 are subgroups of G .
27. Let G be an abelian group, and let n be a fixed positive integer. Show that $N = \{g \in G \mid g = a^n \text{ for some } a \in G\}$ is a subgroup of G .
28. Suppose that p is a prime number of the form $p = 2^n + 1$.
(a) Show that in \mathbf{Z}_p^\times the order of $[2]_p$ is $2n$.
(b) Use part (a) to prove that n must be a power of 2.

29. In the multiplicative group \mathbf{C}^\times of complex numbers, find the order of the elements $-\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i$ and $-\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}i$.

30. In the group $G = GL_2(\mathbf{R})$ of invertible 2×2 matrices with real entries, show that

$$H = \left\{ \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix} \mid \theta \in \mathbf{R} \right\}$$

is a subgroup of G .

31. Let K be the following subset of $GL_2(\mathbf{R})$.

$$K = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \mid d = a, \quad c = -2b, \quad ad - bc \neq 0 \right\}$$

Show that K is a subgroup of $GL_2(\mathbf{R})$.

32. Compute the centralizer in $GL_2(\mathbf{R})$ of the matrix $\begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$.

Note: Exercise 3.2.14 in the text defines the centralizer of an element a of the group G to be $C(a) = \{x \in G \mid xa = ax\}$.