3.3 Constructing Examples

The most important result in this section is Proposition 3.3.7, which shows that the set of all invertible $n \times n$ matrices forms a group, in which we can allow the entries in the matrix to come from any field. This includes matrices with entries in the field $\mathbf{Z}_p$, for any prime number p, and this allows us to construct very interesting finite groups as subgroups of $\mathrm{GL}_n(\mathbf{Z}_p)$.

The second construction in this section is the direct product, which takes two known groups and constructs a new one, using ordered pairs. This can be extended to n-tuples, where the entry in the ith component comes from a group G_i, and n-tuples are multiplied component-by-component. This generalizes the construction of n-dimensional vector spaces (that case is much simpler since every entry comes from the same set).

SOLVED PROBLEMS: §3.3

16. Show that $\mathbf{Z}_5 \times \mathbf{Z}_3$ is a cyclic group, and list all of the generators for the group.

17. Find the order of the element $([9]_{12}, [15]_{18})$ in the group $\mathbf{Z}_{12} \times \mathbf{Z}_{18}$.

18. Find two groups G_1 and G_2 whose direct product $G_1 \times G_2$ has a subgroup that is not of the form $H_1 \times H_2$, for subgroups $H_1 \subseteq G_1$ and $H_2 \subseteq G_2$.

19. In the group $G = \mathbf{Z}_{36}^\times$, let $H = \{[x] \mid x \equiv 1 \pmod 4\}$ and $K = \{[y] \mid y \equiv 1 \pmod 9\}$. Show that H and K are subgroups of G, and find the subgroup HK.

20. Show that if p is a prime number, then the order of the general linear group $\mathrm{GL}_n(\mathbf{Z}_p)$ is $(p^n - 1)(p^n - p) \cdots (p^n - p^{n-1})$.

21. Find the order of the element $A = \begin{bmatrix} i & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -i \end{bmatrix}$ in the group $GL_3(\mathbf{C})$.

22. Let G be the subgroup of $\mathrm{GL}_2(\mathbf{R})$ defined by
$$G = \left\{ \begin{bmatrix} m & b \\ 0 & 1 \end{bmatrix} \middle| \, m \neq 0 \right\} .$$
Let $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$. Find the centralizers $C(A)$ and $C(B)$, and show that $C(A) \cap C(B) = Z(G)$, where $Z(G)$ is the center of G.

23. Compute the centralizer in $GL_2(\mathbf{Z}_3)$ of the matrix $\begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix}$.

24. Compute the centralizer in $GL_2(\mathbf{Z}_3)$ of the matrix $\left[\begin{array}{cc} 2 & 1 \\ 1 & 1 \end{array}\right]$.

25. Let H be the following subset of the group $G = GL_2(\mathbf{Z}_5)$.

$$H = \left\{ \left[\begin{array}{cc} m & b \\ 0 & 1 \end{array}\right] \in GL_2(\mathbf{Z}_5) \,\middle|\, m, b \in \mathbf{Z}_5,\ m = \pm 1 \right\}$$

(a) Show that H is a subgroup of G with 10 elements.

(b) Show that if we let $A = \left[\begin{array}{cc} 1 & 1 \\ 0 & 1 \end{array}\right]$ and $B = \left[\begin{array}{cc} -1 & 0 \\ 0 & 1 \end{array}\right]$, then $BA = A^{-1}B$.

(c) Show that every element of H can be written uniquely in the form A^iB^j, where $0 \le i < 5$ and $0 \le j < 2$.